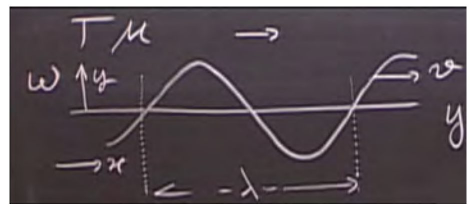
Massachusetts Institute of Technology OpenCourseWare

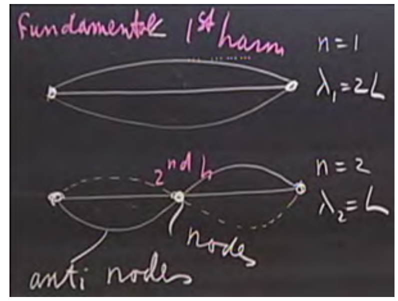
8.03SC **Fall 2012**

Notes for Lecture #8: Traveling & Standing Waves

The last lecture covered normal modes in systems with a finite number of particles and traveling waves in a continuous medium. Now the two ideas converge as normal modes in continuous media are considered. One approach could be to take the results from N beads on a string and let N go to infinity. This is easy to envisage qualitatively (**1:20**), but for the quantitative aspect an approach based on traveling waves and boundary conditions will be used. A few definitions are needed. The *wavelength* λ is the distance the disturbance travels in one oscillation time, as it propagates with speed $v = \sqrt{T/\mu}$ as described by $y = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$. Note that this is a solution of the wave equation since *any* function $f(x \pm vt)$ is a solution. This particular function describes a periodic wave as opposed to just a pulse. If, for a fixed time t, one advances by a distance x in the direction of propagation of the wave, the disturbance will change as a sinusoid. If, instead, one sits at a point and lets the wave move past as time t increases, the pattern repeats after one period P (T was used for some other periodic motions, but here we want to retain T as the tension). Since the wavelength λ is the distance traversed by the wave moving at a speed v in a time of one period P we know that $\lambda = vP$. If ω is the oscillation rate in radians per second, the period must be given by $P = 2\pi/\omega$ and so $\lambda = vP = 2\pi v/\omega$ and $\omega = 2\pi v/\lambda$. We can introduce the *wave number* $k = 2\pi/\lambda$ so that $\omega = kv$ (**4:30**). Use of k gives a more symmetric wave equation: $y(x,t) = A\sin(kx - \omega t)$.



We now consider two identical waves propagating in opposite directions: $y_1(x,t) = A\sin(kx - \omega t)$ and $y_2(x,t) = A\sin(kx + \omega t)$ which we want to add. The sum of any two sine functions is the sine of half the sum of the arguments times the cosine of half the difference. So, the sum of these two waves is simply $y = y_1 + y_2 = 2A\sin(kx)\cos(\omega t)$. The importance of this result is that the spatial and temporal oscillations are now completely decoupled (**7:45**). The overall pattern does not move! It is called a *standing wave.* We now consider fixing the ends of the string, $y = 0$ at $x = 0$ and at $x = L$. At $x = 0$, this is automatic for sine functions. At $x = L$, it happens if $kL = n\pi$, or $k_n = n\pi/L$, where k_n denotes wavenumber for mode n. Furthermore, $\lambda_n = 2\pi/k_n$ and $\omega_n = n\pi v/L$. The frequency in Hz is then $f_n = \omega/2\pi = nv/2L$ (**10:30**). These normal modes have integral numbers of half sinusoids, oscillating at the angular frequency ω_n (i.e. frequency f_n).

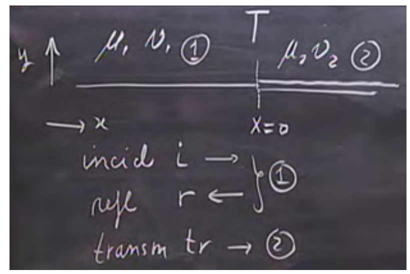


One half wavelength between the fixed ends is called the *fundamental* or *1st harmonic*. A full wavelength is the *2nd harmonic*, etc., and the frequencies are linearly proportional to the mode (harmonic) number, i.e. $\omega_n = n\omega_1$ (**14:10**). In music, the harmonics are referred to as overtones. In quantum mechanics, similar waves play an important role and the modes are called eigenstates. Our general solution for the n^{th} mode is $y_n = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$ (**16:30**). By driving at appropriate frequencies, the resonances corresponding to the normal modes can be easily excited, as has been demonstrated many times, and is done here for the modes on stretched strings.

The same considerations apply to sound (**23:30**), which is a (slight) excess or decrease of pressure with regard to the ambient pressure, and is a longitudinal oscillation. To emphasize this, the variable is changed: $\xi_n = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$. The velocity is always close to 340 m/s for sound near ground level, and this value is used in $\omega_n = n\pi v/L$. If, instead of displacement, we talk about pressure, then the boundary conditions dictate that the pressure change is maximum at the ends and $p_n(x,t) = P_n \cos\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$ (**28:00**). In the demo, the preferred frequency would be 803 Hz to match the MIT Oscillations and Waves class which is 8.03, but actually 2409 Hz had to be used. Then $\lambda = v/f = 340/2409 \approx 0.14$ m. In the demo, the nodes (no sound) are very clear (**34:30**). The speed of sound can be extracted very accurately since f is very precisely known and the $\lambda/2$ distance between nodes can be easily measured (**37:00**).

Energy is moved by waves despite the fact that no mass moves in the direction of the wave (even in a longitudinal wave the particles do not move far, they oscillate about their equilibrium position). We consider initially the kinetic energy (KE). In a short length of the string, for a transverse wave, there is a y velocity and this gives KE of $dE_{kin} = \frac{1}{2}(dm)v_y^2 = \frac{1}{2}\mu dx\left(\frac{\partial y}{\partial t}\right)^2$. Taking the derivative, $\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}A\sin[k(x-vt)] = -Akv\cos[k(x-vt)]$. So the element of KE is $dE_{kin} = \frac{1}{2}\mu A^2k^2v^2\cos^2[k(x-vt)]$ (**40:50**). We obtain the KE over one whole wavelength, by doing an integral: $E_{kin} = \int_0^\lambda \frac{1}{2}\mu A^2k^2v^2\cos^2[k(x_i-vt)]dx$. The constant terms can be moved outside the integral sign to give $E_{kin} = \frac{1}{2}\mu A^2k^2v^2\int_0^\lambda \cos^2[k(x_i-vt)]dx$. The integral is $\lambda/2$ so $E_{kin} = \frac{1}{2}\mu A^2k^2v^2\frac{\lambda}{2}$. With $v^2 = T/\mu$ and $k = 2\pi/\lambda$, in one wavelength $E_{kin} = \frac{A^2\pi^2T}{\lambda}$ (**43:20**). Note that the energy is proportional to the amplitude squared. There is also potential energy in the string stretching to the oscillating shape and it turns out that the potential energy per wavelength is the same as the kinetic energy. Thus, the total energy is $E_{trw} = \dfrac{2A^2\pi^2 T}{\lambda}$, where trw denotes "traveling wave" (**45:20**). A standing wave of a given amplitude is equivalent to two traveling waves of half the amplitude going in opposite directions giving $E_{standing} = \dfrac{A^2\pi^2 T}{\lambda}$ (**50:10**).

In traveling waves, there is energy flow in the direction of propagation. Power needs to be put in. An energy of E_{trw} is injected for every oscillation, i.e. Power$= \dfrac{2A^2\pi^2 T}{\lambda}\dfrac{1}{P}$. But since the period is v/λ, we have Power$= \dfrac{2A^2\pi^2 T v}{\lambda^2}$ (**53:30**). For a traveling wave, this assumes that the power is put in at one end and goes away (maybe on an infinitely long rope). Recall that in the standing wave demo, no energy needed to be put in once the oscillation was going.



In the case of two connected ropes, the tension must be the same (**55:30**). If one rope has mass density μ_1 and the other μ_2, then the propagation velocities are $v_1 = \sqrt{T/\mu_1}$ and $v_2 = \sqrt{T/\mu_2}$. If the junction is at $x = 0$, with the incoming (incident) wave from the left, there will be a reflected wave which must be toward the left, and a transmitted wave toward the right (i.e. from the left like the incident wave). The boundary conditions mean that both y and y' must be continuous at the junction. The incident and reflected waves are in opposite directions but have the same wavelength since they are in the same medium: $y_i = A_i \sin(\omega t - k_1 x)$ and $y_r = A_r \sin(\omega t + k_1 x)$. The transmitted wave is in the second medium so its k is different: $y_t = A_t \sin(\omega t - k_2 x)$. As in all driven cases, the ω is the same everywhere: $\omega = k_1 v_1 = k_2 v_2$ (**1:00:00**). At the junction ($x = 0$), the sine terms are the same for all waves, and so $A_i + A_r = A_t$. Taking the x derivative for each gives a common cosine and so $-A_i k_1 + A_r k_1 = -A_t k_2$. From $\omega = k_1 v_1 = k_2 v_2$ we can replace k with v to get $(A_i - A_r)v_2 = A_t v_1$. Then the ratios $R = \dfrac{A_r}{A_i} = \dfrac{v_2 - v_1}{v_2 + v_1}$ and $T_r = \dfrac{A_t}{A_i} = \dfrac{2v_2}{v_1 + v_2}$ (**1:03:30**). Note that T_r is initially written on the board incorrectly but later fixed.

To consider illustrative cases, first put $\mu_2 = \infty$ which would be a fixed end or wall. Then $R = -1$ and $T_r = 0$, as expected. If $v_1 < v_2$, i.e. $\mu_1 > \mu_2$, $R > 0$ and $T_r > 0$, so what comes back is not inverted, and some non-inverted wave also gets through. This is somewhat like a free-end case. In the case that the two strings are identical, we expect nothing to happen, and indeed from the formulas, $R = 0$ and $T_r = 1$ (**1:08:00**). Leading to the demo, try $v_1 = 2v_2$. Then $R = -1/3$ and $T_r = 2/3$. Also, the wavelength will change in the transmitted wave. Both features are shown in a demo. A free end (open end) can be modeled by $\mu_2 \approx 0$, i.e. v_2 goes to infinity. Then $R = 1$, $T_r \approx 2$ (**1:15:50**). The consequences of this seeming violation of energy conservation are left for the students to ponder.

8.03SC Physics III: Vibrations and Waves
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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.