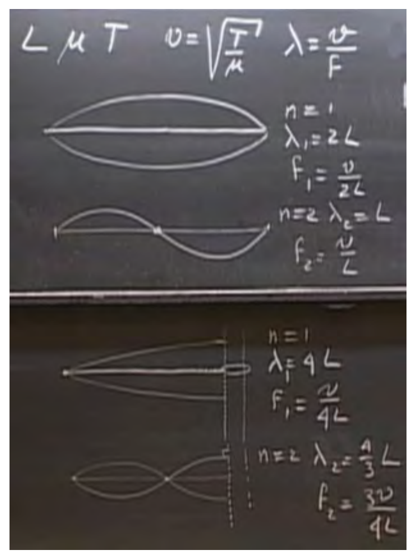
# Massachusetts Institute of Technology OpenCourseWare

**8.03SC** **Fall 2012**

## Notes for Lecture #9: Normal Modes in Sound and Music

Most of this lecture is a discussion of sound and its generation. Moving objects couple mechanical energy into air, setting up pressure waves which are perceived as sound. This can happen both for stringed musical instruments and for sound in an enclosure. The two are very similar except for the boundary conditions. The lecture starts with a reminder of the basic results for strings.

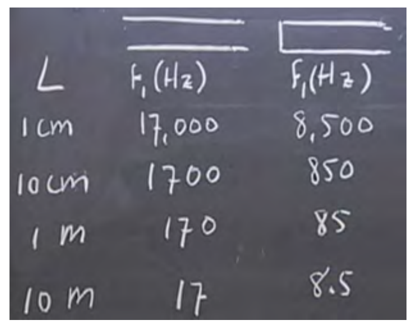
In a string of length $L$ with fixed ends and with a tension $T$ and mass per unit length $\mu$, disturbances propagate with the speed $v = \sqrt{T/\mu}$. From the previous lecture but not repeated in this one, the equation for the mode $n$ in this case is: $y_n = A_n \sin(n\pi x/L)\cos(\omega_n t)$, where the cos term represents the whole string going up and down with time, while the sin term shows the shape of the string at any given time. If $\lambda$ is the wavelength, and $f$ the frequency in Hz, i.e. the number of waves per second, then the speed is $v = \lambda f$. From the boundary conditions, an integral number of half-wavelengths must fit in the length $L$. For the fundamental (or 1st harmonic), $n = 1$ and the wavelength is twice the string length, i.e. $\lambda_1 = 2L$. The frequency is $f_1 = v/2L$. For the second harmonic, $n = 2$, a whole wavelength fits, so $\lambda_2 = L$ and $f_2 = v/L$. In general, $\frac{1}{2}\lambda_n n = L$ or $\lambda_n = 2L/n$ and the corresponding frequency is $f_n = v/\lambda_n = nv/2L$ (**2:40**).




Now different boundary conditions are considered. If one end is free (which is hard to do with strings), we now require the slope to be zero at that end. In this case the fundamental has only one-quarter wave in it, so for $n = 1$, $\lambda_1 = 4L$ and $f_1 = v/4L$. The next possible harmonic has 3/4 of a wavelength, so for $n = 2$, $\lambda_2 = 4L/3$ and $f_1 = 3v/4L$. For each node $n$, $\lambda_n = \dfrac{4L}{2n-1}$ and $f_n = \dfrac{(2n-1)v}{4L}$. Instead of ratios 1, 2, 3, 4 for fixed ends, here we have ratios 1, 3, 5, 7... (**6:30**).

A vibrating object is more efficient in coupling energy to the air if it is attached to a surface, as demonstrated with several sound generators. In musical instruments this is called a sounding board. All string instruments have two fixed ends so the fundamental is $f_1 = \dfrac{1}{L}\sqrt{\dfrac{T}{\mu}}$ (**9:40**). If $L$ is long, the pitch is lower, and vice versa. For higher tension or lower mass per unit length, one gets a higher pitch. In a piano the length and mass per unit length vary a lot, but the tension is about 200 N for each string (about equivalent to supporting 20 kg on each string). For the piano, each string has a unique tone. For other stringed instruments, different notes are generated by varying the length of the string. Some harps have fixed length strings but they can be plucked at different places which changes the mix of harmonics generated.

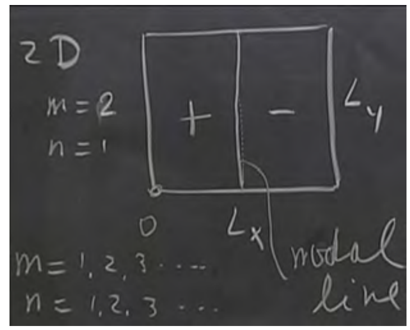
Wind instruments (**18:00**) have boundary conditions depending on whether one or both of the ends is open. If an end is open, then it is a node because it must have the same pressure as the air (the overpressure is 0). For our purposes, the speed of sound is 340 m/s, although the general formula $v = \sqrt{RT\gamma/M}$ is mentioned. Here, $R$ is the gas constant, $T$ is the temperature, $\gamma$ is a ratio of specific heats, and $M$ is the molecular weight. With two open ends, the solutions are the same as for a string, $\lambda_n = 2L/n$ and $f_n = nv/2L$ (**19:30**). By blowing air past the end, normal modes will get excited. A system with one open and one closed end is easy to build and is similar to the (almost impossible to build) case of a string with one end free, so $f_n = \dfrac{(2n-1)v}{4L}$.




This table of frequencies is for open-open (left column) and open-closed (right column) systems (**22:30**). Students are encouraged to calculate these themselves. As expected, large systems have lower frequency. For a 256 Hz tuning fork, the sounding box (if open-closed) should be 33 cm long and this matching is demonstrated. With wind instruments, $v$ is fixed, so the length is the only effective variable. Organ pipes are of various lengths; flutes have holes to change the effective length depending on which hole is covered. For a 16.6 cm open-open instrument, the frequency is about 1000 Hz. If it is closed at one end, the the frequency is only about 512 Hz, as is demonstrated (**31:30**). A tube open at both ends is rotated at various speeds to demonstrate the frequencies of the normal modes. Musical instruments often have many harmomincs active at once and this gives a complex wave pattern. A tuning fork generates very close to only one mode, and a flute is also demonstrated that has a very pure tone. Musical instruments showing more complex patterns are demonstrated by the students.

For transverse waves in a string with two fixed ends separated by a distance $L$, the normal mode solutions to the wave equation given by $\dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2}$ are $y_n = A_n \sin(n\pi x/L)\cos(\omega_n t)$, where $\omega_n = vk_n = vn\pi/L$ (**54:50**). For longitudinal waves such as sound, other boundary conditions are possible. If it is open-open then the solutions for overpressure $p$ are exactly the same as for displacement $y$ in the transverse case with two fixed ends. For open-closed, one gets instead solutions similar to those for one fixed and one free end, namely $k_n = \dfrac{(2n-1)\pi}{4L}$ but still $\omega_n = vk_n$.

This can be generalized to two dimensions where the wave equation is $\dfrac{1}{v^2}\dfrac{\partial^2 z}{\partial t^2} = \dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$ (**58:00**), where $z$ is the vertical displacement of a surface above the $x - y$ plane. The analog of fixed end boundary conditions in a string is to fix the surface all around the edges. This makes a vibrating rectangular membrane, rather like a rectangular drum. If the lengths are $L_x$ and $L_y$, then the boundary conditions are that $z = 0$ at $x = 0$ and $x = L_x$, and $z = 0$ at $y = 0$ and $y = L_y$. It is easy to show that a multiplication of spatial parts very similar to those for a string solves the wave equation and boundary conditions: $z(x, y, t) = A_{m,n} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{n\pi y}{L_y}\right) \cos(\omega_{m,n} t)$ where $\omega_{m,n} = vk_{m,n} = v\sqrt{\left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2}$ (**1:00:30**).




There are many possible normal modes, of which the lowest has $n = m = 1$, for which the whole membrane moves up and down at the same time. For $m = 2$, $n = 1$, there are two oppositely moving regions, separated by a nodal line (where the amplitude is 0) at the midpoint $x = L_x/2$. The next mode up would have four regions. Nodes are demonstrated by driving a plate with powder on it. At the nodes, the plate is stationary and powder can simply sit in place, elsewhere it is vibrated away (**1:04:45**). However, notice that the boundary conditions are not what was discussed above. In this case, the middle is vibrated and the outer edges are free. Such plates are called Chladni plates. The modes are more complex than those for a surface fixed at the edges. The change in modes with frequency can be quite dramatic.

We can now consider the three-dimensional wave equation (**1:08:30**). The case of a sound cavity of lengths $a$, $b$, $c$ along the $x$, $y$, $z$ directions with closed sides and open ends in the $z$ direction is considered. The wave equation is: $\frac{1}{v^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$ with solution:

$$p_{l,m,n} = P_{l,m,n} \cos\left(\frac{l\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{c}\right) \cos \omega_{l,m,n} t$$

where $l$=0, 1, 2,... , $m$=0, 1, 2,... , $n$=1, 2, ... (**1:11:00**). Note that the condition on $n$ is different since $n$=0 would just be no motion at all. Here $\omega_{l,m,n} = v\sqrt{(l\pi/a)^2 + (m\pi/b)^2 + (n\pi/c)^2}$. If $c$ is the biggest dimension, as is typically true for musical instruments, the lowest mode is $\omega_{0,0,1} = v\pi/c$, or $\omega_{0,0,1} = v\pi/L$ if we call the length of the instrument $L$ as was done previously. For typical instrument dimensions, the next few lowest modes also involve varying the last index $n$ (**1:14:30**).

A demo showing the effect of the difference in the speed of sound in air versus helium follows ($v$ in helium is 3 times that in air according to the formula given above).

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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.