8.03SC Physics III: Vibrations and Waves, Fall 2012
Transcript – Lecture 10: Review and Discussion of Problem Solving

PROFESSOR: I want to start with the physical pendulum, which is exactly the same one that I discussed during my first lecture. This is a hoop. It has mass m and radius R. And we were calculating the period of this hoop as it oscillates. And we did that using the famous torque equation from 8.01. The torque relative to point P is the moment of inertia relative to point P times the angular acceleration alpha.

Today, I will do this again, but I will use the conservation of energy to show you that in case there is no damping, when mechanical energy is conserved, that you can find the correct differential equations through the conservation of energy. If the thing is swinging, in general, there are two components. You have kinetic energy, and you have potential energy.

And the kinetic energy, K, is 1/2 times the moment of inertia about point P times omega squared. You remember that from 8.01. And this omega is theta dot. We call that the angular velocity. This angular velocity changes with time. When the object goes to equilibrium, the angular velocity is at maximum, and when the object comes to a halt, the angular velocity is 0.

Do not confuse this omega with omega 0. I will give that as 0 now to distinguish it from omega, which is the angular frequency. The angular frequency is a constant of the motion, and that is 2 pi divided by T0 if T0 is the period of oscillation. So this is the kinetic energy, and this is the square of the angular velocity.

And then we have potential energy. Let this be point A, and when we are here, the center of mass is at point B. And so the potential energy u-- or actually, I should say the potential energy at point B minus the potential energy at point A-- it's always the difference in potential energy that matters-- that equals mgh, h being the difference in height between point B and point A. So this here is h-- mgh-- Massachusetts General Hospital. That's the way to remember it.

Now, h is very easy. h is the same as R times 1 minus the cosine of theta. We went through that many times, so you can easily confirm that. R is the radius of this circle. This is the radius.

Now, for small angles, the cosine of theta is theta squared divided by 2. So I can rewrite this differential equation now that E total--

AUDIENCE: [INAUDIBLE]?

PROFESSOR: I see nothing wrong with that. I'm sorry. Is there anything wrong with it?

AUDIENCE: I thought the cosine was 1 minus theta squared over 2.

PROFESSOR: The cosine's theta alone-- you're right, is 1 minus theta squared over 2. Thank you very much. Thank you. So I'm going to write it now as the total energy is 1/2 Ip times theta dot squared. And then I get plus mgR times theta squared over 2. And what you do now, since this is a constant of the motion, you always do that if you work with the conservation of energy. You take the time derivative of this equation, which must be 0, because mechanical energy is conserved, and out pops the differential equation that we also found during my lecture number one, when I used the different method.

So I take the time derivative-- so this T eats up this 1/2, and so we get I of p times theta dot times theta double dot-- I have to use the chain rule-- and then I get plus mgR. This 2 eats up this 2, and so I get theta times theta dot, and now this equals 0, because the EDT is 0. And whenever you do that, you will always see that the theta dot term disappears, or if you have the equation in x, then the x dot term disappears. And you see that. This term goes. You can divide it out.

And so your differential equation takes on, now, a very familiar form. Let me write this 2 here. So I get theta double dot plus mgR divided by I of p times theta equals 0. And this differential equation, you should recognize the solution to that equation is immediately obvious, omega 0 squared. I use now the omega 0 equals mg times R divided by I of p.

So the general solution for this oscillation then becomes that theta is theta 0-- that is the amplitude-- times the cosine-- or the sine, if you prefer that-- omega 0 t plus some phase angle phi. That is the general solution, and if you knew the initial conditions, what the situation was at t equals 0, if you know the angular velocity at t equals 0, and if you know where it is, t equals 0, you can solve for this theta 0, and you can solve for this phi. But omega 0 is independent of the initial conditions. Very well.

Now I would like to cover a case whereby I'm going to introduce damping. Whenever we deal with damping, there are two terms that are important in physics. That is the speed itself, there is a damping term, which opposes the velocity, which is linearly proportional with the speed. And there is a damping term which opposes the velocity, which is proportional with the square of the speed. We will always leave the square out with 8.03, because the differential equations become impossible to solve. Maybe numerically you can do it, but not analytically.

However, if you're interested in all the physics, which is wonderful, with the v square and the v, my lecture number 12 on OCW-- OpenCourseWare-- from 1999, Newtonian Mechanics, I deal with the v square and with the v term, and I do many demonstrations to show you that there are certain domains where the v term is important. Recall that the viscous term, and there are certain domains in physics, where the v squared term is important.

So I will now simply restrict myself, then, to the damping force, which is linearly proportional with the velocity, and we will write it down as F equals minus b times v. We will use a shorthand notation that gamma equals b over m. That is only a shorthand notation. I will erase this. We don't need this anymore. We need so many blackboards today that I will use this blackboard for my damping problem.

When you deal with damping, we recognize three different domains. One domain, whereby gamma is smaller than omega 0, we call that under-damped. Then we have a domain whereby gamma is larger than omega 0. We call that over-damped. And then you have a very special case where gamma equals omega 0.

And the behavior of these three different kinds is very different. I will only discuss with you today the under-damped case. So I have a pendulum, and this pendulum has mass m and length l, and I will assume that the mass in the string is negligibly small. I have damping, and this is the case that gamma is smaller than omega 0.

This is the equilibrium position of the pendulum. I will therefore call this position x, and I'm going to put all the forces on this object in this picture, which is mg, and that is T. And there are no other forces on that mass. Now, if we deal with small angles, as we have done before more than once, then the tension T is very close to mg.

So the only force that is driving it back to equilibrium, the only restoring force, then, is the horizontal component of T. So that is this horizontal component. That's the only one that we are concerned about. So the differential equation, then, in terms of Newton's Second Law, becomes mx double dot. And then we get minus T times the sine of theta minus b times x dot. That's the damping, and this is the restoring force due to the tangent in the string. And the sine of theta is x divided by l.

So I get mx double dot plus mg divided by l times x-- that is the sine of theta-- plus b times x dot equals 0. Make sure I have this, m doubled dot. I have the plus sign here. I have mg x over l. That's fine. So now I divide m out, and I also use the shorthand notation that omega 0 squared is g divided by l. These are shorthand notations, which give you a little bit more insight when you see the solutions.

So that gives me, now, the differential equation that x double dot plus gamma times x dot plus-- I divide m out-- plus omega 0 squared times x equals 0. And that is the differential equation that you will recognize. And I would never want you to derive the solution to this differential equation. That's, of course, way too time consuming to do that during an exam. And the solutions to this are given on your formula sheet, which will be part of your exam.

So let me write down here what that solution is. So x, as a function of time, is a certain amplitude times e to the minus gamma over 2 times t. And then we have a cosine or a sine, cosine omega t plus some phase angle alpha. If you knew the initial conditions, then you can find what the amplitude is, and you can find what the angle alpha is. If you don't know the initial conditions, then you do not know this.

There is another way that you can write this form, which is sometimes better. And I cannot tell you when it is better and when it is not better. It depends on the initial conditions. But I want you to appreciate that you can also write this, for instance, as e to the minus gamma over 2 times t and then B times cosine omega t plus C times the sine of omega t.

From the physics point of view, there is no difference. But from the math point of view, there is a difference. You now have these as the two adjustable constants depending upon the initial conditions. And sometimes, if you assume this, it works faster than if you assume that. And as I said, it really depends on the initial conditions which goes faster. But they are very similar, of course.

Omega which is now the frequency which this object is going to-- angular frequency is going to move, that omega is always lower than omega 0. And that shouldn't surprise you, because when there is damping, there is something that is opposing the motion. And so it shouldn't surprise you that when you solve for omega that omega squared is omega 0 squared minus gamma squared divided by 4. That is also something that we would probably give you on the formula sheet, because you will only find that if you substitute this back into the differential equation.

We often-- particularly French-- like to write down q equals omega divided by gamma. This is the quality-- omega 0 divided by gamma. And if you do that, then omega squared can also be written as omega 0 squared times 1 minus 1 over 4Q squared. So you see that if Q is, for instance, 10, which is by no means absurdly high, that omega is only 1/8 of a percent lower than omega 0. They are that close. And even when Q is two, the difference is only 3% between omega and omega 0.

So what happens here is that the amplitude is decaying at a time constant-- the 1 over e time constant, which is 2 divided by gamma seconds. If you put in t, 2 divided by gamma, the amplitude goes down by a factor of e. Since energy is always proportional to amplitude squared, the decay time of the energy is not 2 over gamma but is 1 divided by gamma.

Now I'm going to make a change. I'm going to drive this system. This is no longer the equilibrium position, but the equilibrium position was really here. And I am driving the top of this pendulum with a function eta is eta 0 times the cosine of omega t. So I am driving it now.

So this eta is in terms of inches-- millimeters. This is the motion of my hand. And this omega is no longer negotiable. This omega has nothing to do with this omega. This is the frequency with which the system likes to oscillate. That, now, is the frequency with which I want the system to oscillate. They're totally unrelated. This is my will. This is non-negotiable. I can make these anything I want there. I can make it 0. I can make it large. I can make it, also, that value, of course.

So now the equilibrium position is not here. And I will call, now, this distance from the equilibrium position x. You always use in your coordinate system the equilibrium position, as your 0. So what is the only thing that is going to change now is that the sine of theta is no longer x divided by l, but the sine of theta is x minus eta divided by l, because notice, if this is eta, which is the position of my hands, then the sine of this angle is now x minus eta divided by l.

And this, now, you have to carry through your differential equations. And what you will see, then, if you do that, that this now is not 0, but this now becomes eta 0 times omega 0 squared times the cosine of omega t, my omega. So instead of sine theta x divided by l, all you have to do is this. You know that eta is eta 0 cosine omega t. You carry that through, and you will see that your differential equation now changes.

Notice that x double dot is an acceleration. Notice that gamma times x dot is an acceleration. Gamma is 1 over second, and x dot is meters per second. This is an acceleration. Notice that this is an acceleration. It has a dimension omega squared, which is 1 second squared times meters. That is an acceleration. Notice that this is an acceleration. Eta 0 is meters, and omega squared is 1 divided by seconds squared. So these are apples, these are apples, these are apples, and these are apples. So the equation looks very kosher to me.

Now, the solutions, that's a different story. That is something that I wouldn't want you to derive either, but the solutions now become as follows. I now get x as a function of t. There's an amplitude, which is a very strong function of omega, but I will simply write it down as an amplitude A times the cosine of my omega t minus some phase angle delta. And this is what we call the steady-state solution-- steady state.

And this A becomes, then, a rather complicated function. Upstairs, I get the eta 0 omega 0 squared. I get this part. And downstairs, I get the square root of omega 0 squared minus omega squared squared plus omega gamma squared. And that is the amplitude as a function of omega. And the tangent of delta becomes omega gamma divided by omega 0 squared minus omega squared.

If we plot that function A as a function of omega-- so here is omega, and here, I'm going to plot the A-- then I can recognize that if omega goes to 0, that means if I move this very slowly, that for sure the amplitude of this object must be the same as my hand-- must be at eta 0. If it takes me one week to go from here to here, then of course, the pendulum is always hanging below my hand.

So when omega goes to 0, I can always check the result here that A must go to eta 0. And indeed, if you put in omega equals 0, you will see that is the case. If omega goes to infinity, then A goes to 0. And then there is a very special case that is when omega happens to be omega 0, which is the frequency of the system in the absence of damping. This is my omega 0. It is not this one, but it is the omega 0.

Then you will get an amplitude which is Q times eta 0. And again, I just happened to remember that. If you substitute this back in here-- this omega equals omega 0-- you will see that. And that's always very nice to remember, at that omega 0, you always get Q times more than your driver. Is Q is 100, you'll get 100 times the amplitude of the driver.

So here, you get eta 0 if omega is 0. And then, when you reach that omega 0 frequency, you come out high, and then it goes back to 0, and this point here is Q times eta 0. And we discussed before, I'll never make too much of a deal out of that, that the actual maximum of this curve is not at omega 0 but is always a little lower. That is really more of an algebraic interest than it is of physical interest, because if gamma is low, if Q is high, the two almost coincide.

So this steady-state solution has no adjustable constant under which the system has lost its memory of what happened when we started driving it. So at t equals 0, if we know what x is, and if we know what x dot is, and if we know what the driving term is at t equals 0, then the general solution for all times larger than 0 is the sum of the transient solution and the steady-state solution. And so here, you see the transient solution, you can write it in this form, or you can write it in this form.

And here, you see the steady-state solution. This A has nothing to do with that A. This A follows from the initial conditions. Just like that alpha, this A does not follow from the initial conditions. This A follows from eta 0, what my amplitude is of my driver, and it follows from omega. So the two As don't confuse the two.

So the general solution is then the sum of the two. The transient one will die out. If you wait a few times 2 over gamma, the transient is gone, and so you end up, then, only with the steady-state solution. And you had some chance in your homework to work with that.

If we are not the driving the system, and if we have one object on a spring or on a pendulum or on a floating object in liquid, then there is one frequency, one normal mode frequency, one resonance frequency, we call it also the natural frequency. The moment that you couple oscillators, if you couple two, you get two normal mode frequencies. You get two resonance frequencies. And if you have three objects, you get three normal mode frequencies.

So now I would like to discuss with you a case whereby I couple two oscillators. If I gave you on an exam three coupled oscillators, that would be very nasty, because it's extremely time consuming. If I gave you four coupled oscillators, that would be criminal, because you cannot finish that in 85 minutes. So two is certainly within reason. Three is marginally within reason. Four is out of the question.

When we have coupled oscillators, we always leave damping out. And yet, we will learn a lot from it. Even without damping, we will learn a lot. So what I have chosen to do with you is a spring system with two objects, two masses, and the springs have no mass, negligible mass. The spring constants are k, and the masses of the objects are m. So this is the equilibrium position of these two objects, and the ends are fixed of the string.

I will introduce a shorthand notation that omega 0 squared equals k over m. Now what I do, I offset both these objects from equilibrium just as I did that here. I always offset them in the same direction, and I call that positive. Is that a must? No. Is that useful? Yes. So I offset them.

So this position, now-- this is object number one, and this object number two-- this is now a distance x1 away from equilibrium, and this one is now at a position x2 away from its equilibrium. You always want to know what the displacement is relative to its own equilibrium. And its own equilibrium for 1 is here, and the equilibrium for 2 is there.

I now make the following assumption that x2 is larger than x1. Is that a must? No. Is it useful? Yes. But if you want to assume that x2 is smaller than x1, be my guest. I will assume that x2 is larger than x1. Now, follow me closely now. So we have an object here, and we have an object here. And this is now one spring. This is the other spring, and this is the third spring.

It is immediately obvious that this spring is longer than it wants to be. So there is a force that drives it back to equilibrium. If x2 is larger than x1, this spring is also longer than it wants to be.

So it wants to contract, so there is a force in that direction. If this spring is longer than it wants to be, it wants to contract, so there is a force on x2 in this direction. This spring is clearly shorter than it wants to be, so it's pushing, so there is a force in this direction.

So now, I'm going to write down the differential equation first for object number one. So I've got n x1 double dot. So this one is minus k times x1, and this one is plus k times x2 minus x1. Is it a disaster if it turns out that x2 is not larger than x1? Not at all. This equation now is correct for all situations. The fact that I have assumed that x2 is larger than x1 gave me a plus sign here. So my differential equation is safe no matter what x1 is relative to x2. So you can always make that assumption, and you don't have to worry later anymore about signs.

I can divide, now, m out, and then I get that x1 double dot plus 2 omega 0 squared times x1. Well, you notice I have one here, and I have one here that have both the minus sign. And then I have here plus sort of becomes minus omega 0 squared times x2. That is 0. So that is my first differential equation. So I put a 1 here.

So now I go to the second one. I get mx 2 double dot. Now I have two forces, both in the negative direction. First, I have the one that drives this one away from equilibrium. So I get minus k times x2 minus x1.

And then I have the force due to the fact that this one is shorter than it wants to be by an amount k2, so I get minus k times x2. It's shorter by an amount x2 than it wants to be. So I can divide m out, so I get mx2 double dot, and then I get plus omega 0 squared times x2. And then I get minus omega 0 squared times x1 equals 0.

Now, compare this one with this one. Notice the incredible symmetry. I could have found this one by changing a 1 to 2 here and by changing a 2 to 1. Of course, the system is so symmetric--did I--

AUDIENCE: [INAUDIBLE].

PROFESSOR: Yes, thank you very much. For the 2, right? Yeah, thank you. The system is so symmetric that clearly, nature cannot make any distinction between 1 and 2. So it is in this particular case, because of the beautiful symmetry, it is obvious that these two differential equations look very, very similar.

Now I'm going to make an important step. I'm going to substitute in here x1 is C times cosine omega t, and x2-- this is C1-- is C2 times cosine omega t. I want to know what the normal mode frequencies are for this system. I want to solve for omega. I want to find omega. I'm not driving the system.

Since I have no damping, in which case at normal mode solutions, either of the two objects are in phase with each other, or they're out of phase with each other, but if they're out of phase, we can always take care of that with a minus sign. So I'm now going to substitute that in here. And I may want to continue working on the center board, in which case I might as well erase this so that it stays a little compact.

So I'm going to substitute this trial function into my differential equations. And every term will have a cosine omega t, so I dump all the cosine omega t's. So I go to the equation, which has a big 1 there, and so that 1 becomes C1. Now, x1 double dot gives me a minus omega squared, right? If I take cosine omega t, and I do the second derivative, I get minus omega squared in front of it.

So I get C1 times 2 omega 0 squared. That is that 2 omega 0 squared. And then I get minus omega squared. And then I get minus omega 0 squared times C2, and that equals 0.

And then I go to my second differential equation, which is this one. But I'm going to rearrange the C1s and the C2s. So I'm going to put the C1s here, so I'm going to get minus omega 0 squared times C1. And here, I'm going to get plus 2 omega 0 squared minus omega 0 squared times C2, and that equals 0. So I can leave this plus out.

So what do I have here now? I have here two equations with three unknowns. I want to know what omega is in the normal modes, but I also would like to know what C1 and C2 is. Well, that is tough luck. You can't have it both ways. We do not know at all what C1 and C2 is separately.

But you can always find, in the normal modes, without knowing anything else what the ratios are of those amplitudes C1 and C2. And you can always solve for omega, and you will see how that works. So with two equations and three unknowns, you can only solve for omega and for the ratio C1 over C2.

So now you may do that any way you want to. This is not so difficult to solve this. I will, however, use Cramer's rule, and I will introduce a D, which is the determinant of the following: 2 omega 0 squared minus omega squared. And then I have here minus omega 0 squared. And then I have here minus omega 0 squared. And here, I have 2 omega 0 squared minus omega 0 squared.

And this, now, must be 0 if I am looking for solutions which are not C1 is 0 and C2 is 0. It's clear that if C1 and C2 is 0 that your equations are satisfied. 0 is 0. But that's not an interesting case. And the only way that you can get a solution which is interesting, with values for C1 and C2 which are not 0, is by making this determinant 0.

So that means that D, which now becomes 2 omega 0 squared minus omega squared squared minus omega 0 to the 4th, you have to make that 0. And when you do that, you find two values for omega. You find that omega squared is 2 omega 0 squared plus or minus omega 0 squared. Those are the two solutions.

And when we evaluate those two solutions, you find the result, which is so embarrassingly simple that you could almost have said that without any work-- almost. Notice that omega minus, which is the lowest one of the two, is the same as omega 0, because that's the minus sign.

What does it mean if omega minus is omega 0? Well, it means, of course, that the two objects are just oscillating like this. The inner spring is never stretched, is never longer than it wants to be, is never shorter than it wants to be. So each one is only driven, so to speak, by the outer spring.

So that's immediately obvious that that's the normal mode. And you can make the prediction that C1/C2 must be plus 1. They must go in unison like this. And you can confirm that by substituting this omega 1. If you substitute that in this equation, you will find the ratio C1 over C2. If you prefer this equation, be my guest. You can do that, too. And you will find no matter which one of the two you take that it is plus 1.

Now, omega plus is less obvious. It's the square root of 3 times omega 0. That's when you take the plus. Even though it's not obvious that it is the square root of 3, it is clear what is happening. You added two objects, and now they're doing this. They are exactly 180 degrees out of phase with each other. So now you can make comfortably the prediction that C1 over C2 is minus 1. And indeed, if you substitute the square root of 3 omega 0 in either this or in that equation, this, indeed, comes out.

So the general solution, now, for a given initial condition-- so if now I give you the initial condition-- then the general solution for x1 would be x0 minus-- the minus makes reference to that lowest frequency-- times the cosine. Or we got minus t plus some phase angle minus. All these minus signs make reference to this mode. And then I have plus x0 plus times the cosine of omega plus t plus some phase angle plus. Of course, if you prefer, instead of cosine, sines, that's fine, of course.

And you have four adjustable constants, one, two, three, four. And if you know the initial condition, if you know what x1 is and what x1 dot is and what x2 is and what x2 dot is at time t equals 0, you can solve for all four, in principle. There is no longer any freedom for number 2, because all four adjustable constants have now been consumed. So therefore, you can write down out a solution for x2 by simply making this one a 2. And this whole term here is the same. The only difference is that this plus sign now becomes a minus sign. Nothing else is different.

The frequencies are the same, otherwise they wouldn't be normal mode frequencies. And it is the ratio here of the C1/C2 that is plus 1, and it is the ratio here that is the minus 1. That's the reason why this becomes a minus and why the plus here remains a plus. So that, now, is the general solution if you also know the initial conditions.

So now we're going to drive this system. So now we go back to where we were, and I'm going to drive this end. And I'm going to drive this end with my hand. This is eta, and eta equals eta 0 times cosine omega t. There is only one term in all of these on the blackboard that is going to change. And that is this term. This spring here on the left is no longer shorter by the amount x1 but is shorter by the amount x1 minus eta.

So it's only that term-- it's only this minus kx1-- that now have to be changed into minus k times x1 minus eta. Nothing else changes. But that has major consequences, of course. For one thing, if you're going to substitute now these trial functions, omega is no longer negotiable. Omega is my omega now. I set this omega. You're not going to solve for that omega. That would be an insult to me. I dictate what omega is.

So that means I now get two equations with two unknowns, C1 and C2. Omega is no longer unknown. So now I get solutions for C1, and I get solutions for C2. And of all the equations on the blackboards that you have, the one that is going to change is the first one. And when you carry through that eta, which is eta 0 cosine omega t, this 0 here changes into eta 0 times omega 0 squared. The cosine omega t is gone, because I've divided cosine omega t out in all those other terms.

So now I use Cramer's rule, and now I can actually come up with a solution for C1, no longer ratio C1/C2. No, I can actually come up, now, with solution. So C1, now, using Cramer's Rule, this now is my first column, eta 0 omega 0 squared 0. And this now becomes my second column, minus omega 0 squared, and I get here 2 omega 0 squared minus omega squared divided by D.

If I pick a random value for omega, D is not 0. D is only 0 at those two resonance frequencies. And then C2 becomes-- the first column is like this. That is 2 omega 0 squared minus omega squared minus omega 0 squared. And now the second column becomes eta 0 omega 0 squared and a 0 divided by D.

If I work C1 out a little further, then I get C1 equals eta 0 omega 0 squared times 2 omega 0 squared minus omega 0 squared divided by D. And here, I get that C2. This is 0, so I get plus omega 0 to the 4th times eta 0 divided by D. And D, now, is this determinant that is not 0. It's only 0 for those two special frequencies.

If you want to see what the amplitudes now are as a function of omega-- it's a very, very interesting behavior-- then what helps is you go omega first to 0, and then you see what happens. And it's not so intuitive what happens. If you put in omega equals 0, you'll find that C1 is plus 2/3 eta 0. You can confirm that by substituting that into C1. You will see that that's what happens. And you'll find that C2 is plus 1/3 eta 0. So that is at omega equals 0,

And when you go to infinity with frequencies, then it should not surprise you that C1 is 0 and that C2 is 0. And then there is one case which is pathetic. And that is the case that I make omega squared, 2 omega 0 squared. At that frequency, C1 becomes 0, but C2 is not 0. And I spent almost the whole lecture with you on that demonstrating that in three different ways that indeed, there is this bizarre solution.

So when omega squared equals 2 omega 0 squared, then C1 becomes 0, but C2 is not 0. So now you can make a plot of C's as a function of omega. And during that lecture, I showed you three of those plots, and I used the convention, then, that when the object is moving in phase with the driver, I put it positive and out of phase, negative. And I plot here C divided by eta 0. And I will use a color code. I will do C1 in red, and I will do-- C1 is red, and C2 I will do in white chalk.

So here is 0 omega. Here is omega 0, which is my omega minus. And then my omega plus was at 3-- the square root of 3. So that's about 1.7. So here is my omega plus. And then things go nuts. That is when D goes to 0.

So C1 is 2/3 here. And then it goes to infinity there. And then here, when it is 1.4 times omega 0, the square root of 2, then it goes through 0, and it comes up here and then I get a curve here, without being too precise. And for C2, I get 1/3, and then it goes up. And then C2, something like this, and then I get something like that. Yeah, I can live with that.

Now, we ignored damping. And because we ignored damping, we get these un-physical infinities when you hit the frequency omega minus and omega plus, these resonance frequencies. Now, if you include damping, then the solutions become extremely complicated. But of course, you avoid the infinity values. But the resonance amplitude can still be very high. So these plots are still very useful provided that you don't interpret infinities is as being real but something that is very large. If Q is high, then of course, the amplitudes are enormously high.

And this can lead to destruction. We've seen the movie of the Tacoma Bridge. And then we have seen the dramatic experiment of the breaking wine glass. You remember that wine glass demonstration. Because of the catastrophic success, to use Bush's words, of that demonstration, students have asked me for an encore. They would like to see it again.

And that's a very reasonable thing to do. It fits very nicely into this concept of coupled oscillators. A wine glass would have a huge number of oscillators coupled. And a wine glass-- I have one here-- can be made to oscillate easily in its lowest normal mode. I have to wash my hands to make you listen to it, because my hands are now a little greasy from the chalk.

And this is the frequency. That is the lowest mode. If it is a circle, like this, looking from above, it becomes an ellipse, and then it becomes an ellipse like this, and the ellipse like this, and the ellipse like that. And if now you drive that with sound at exactly that frequency, and you put in enough power-- another way of saying is if you make eta 0 large enough-- then the system can break.

It's making a lot of noise. I warn you. Those who are sitting here, I really think you should protect your ears. The sound can be deafening, so be careful. And those who are a little bit further away, make sure that you close your years. I have the luxury that I can turn my hearing aids off. But without my hearing aids, believe me, I can still hear a lot, so I also will have to protect myself. So one hearing aid is off, the other is off, and I'm going to put this on anyhow.

I'm going to stroke this glass with a frequency which is only slightly different from this frequency of the sound. That is about 427 hertz, which you can very easily hear. And then you will see the glass in slow motion because of the fact that the stroboscopic light has a slightly different frequency than the-- that should be coming up now.

AUDIENCE: Yeah, it will.

PROFESSOR: And you say it will. Thank you very much. I'm going to make it very dark now, because this is important that you can see this very well. So we'll make it completely dark in the room. I'm going to protect my ears. I hope you can still hear me. I can hardly hear myself. And I'm going to drive the system now at very low amplitude at a frequency close to its resonance.

You can already see that the glass is moving. I'm going to increase the amplitude. You see it moving? I can go a little bit over the resonance and under the resonance. You will see it stop moving then. So I do that purposely now. Now I'm off resonance. I'm over it. And now I'm off resonance. I'm under resonance. I'm below. And now it's 423 hertz, but the resonance is at 427. So I'm going to put it back at 427.

And now I'm going to increase the sound. So I warn you. And that's it. It broke. And it broke very fast. So this is an ideal moment for a break. Since you had such a good time with the breaking glass, let's settle for four minutes. We'll reconvene in four minutes.

[BLOWS WHISTLE] OK. I now want to discuss with you continuous media. If you have n coupled oscillators, then you get n normal modes. When you make n infinitely high, then you get continuous media. And if we have one-dimensional continuous media, like a string or a pipe with air-- sound-- then we have studied the transverse motion of a string. And then we derived that now you have to use the wave equation. We derive the wave equation. And then you get that d2y / dx squared is 1 over v squared times d2y / dt squared, y being now the displacement away from equilibrium in this position if x is in this direction.

AUDIENCE: [INAUDIBLE].

PROFESSOR: And yeah-- that is fine.

AUDIENCE: [INAUDIBLE].

PROFESSOR: And so-- excuse me? Ah, it's amazing how you can look at it and think that it's fine, and it is not. All right, so v, in the case of a string, is the square root of t over mu. That was just a special case for the string. We also examined longitudinal motion, again 1D. Sound is a longitudinal wave. And if you deal with pressure-- you think of sound as being a pressure wave-- then you get d2p dx squared equals 1 over v squared times d2p dt squared.

P, then, being positive, would be overpressure, over and above the ambient one atmosphere. And if it's negative, then it is below. So it's not the total pressure, but it is the overpressure. And v, then, is the speed of sound, which in air, room temperature, is about 340 meters per second.

If you prefer not to work in pressure but in terms of the actual position of the air molecules in analogy with the position of the string, then you can write down the same equation in terms of xi, or d2 xi dx squared 1 over v squared d2 xi dt squared, that, then, gives you the position, the actual motion, of the air molecules. I often prefer the pressure, and I will follow that also today.

So there are an infinite number of normal modes. And the ratios of the amplitudes of two adjacent oscillators-- they are coupled now-- reflects itself, of course, in terms of the overall shape, which you can best see when you deal with a string. As I said, it's easiest to see with transverse oscillations what the displacements look like. It's harder to see that with sound.

I also discussed with you a special situation that I connected to media-- medium one and medium two. And I gave these two media different masses per unit length. When you set up a traveling wave on a string, or you set it up-- you could set up a pulse-- it reflects at the end. And how it reflects depends on the boundary conditions at the end. And when you connect them, two media, then you get not only a reflection, but you also get some of the pulse. Some of the wave goes into medium two.

So let's assume that we have an incident wave coming in like this, and we have here mu 1, and we have here v1 given by this equation. They both have the same tension, T, and here we have mu 2, and you have v2. We discussed that. We used the wave equation to solve what happens when we have an incident harmonic wave coming in. We derived even this speed of propagation using the wave equation. None of this came out of the blue. We always derived that.

And then we found by using the boundary conditions at the junction, namely that the string is not breaking, and that dy dx on the left side is the same as dy dx on the right side with no boundary conditions, we found that the amplitude of the reflected wave-- and the same would hold for a pulse-- divided by the amplitude of the incident wave was v2 minus v1 divided by v1 plus v2. And I called that R-- reflectivity.

And we found that the amplitude of the transmitted wave, or pulse, divided by the amplitude of the incident 1 was 2v2 divided by v1 plus v2. And I called that shorthand notation transmitivity. And when we had done that, I put in some simple test cases where our intuition is very good. The first thing I did was, suppose mu 2 is infinitely high. So mu 2 is infinitely high. In other words, that medium two is a wall.

That means the string number one cannot move. It's fixed at the end. So that means v2 is 0. And we go to this equation, and we find that R equals minus 1. v2 is 0. You get minus v1 over v1. And we like that because what it means is that when a mountain rolls in, it comes back as a valley. And when a valley rolls in, it comes back as a mountain. And I demonstrated that with strings.

It was very pleasing that T of R is then 0. Well, it better be 0, right? If everything comes back at you-- upside down, but nevertheless, everything comes back at you, you expect that nothing goes into the wall. And you see when v2 is 0 that TR is 0. And we were all very happy, and we could all sleep.

But then-- and you guessed it-- then I said, let's suppose mu 2 becomes 0. So I attach that string to nothing-- to empty space. Is that practical? Yes. I can do it. I can take a nickel wire, and I have here a magnetic field-- very strong-- and I can pull on that nickel wire, and the end of the nickel wire ends up in nothing, in empty space, but I keep the tension on. So it's completely practical. It can be done. It's not just notes.

So mu 2 goes to 0. That means that v2 goes to infinity. And then we looked at R, and we say, well, if v2 goes to infinity, then R equals plus 1. And we were all very happy. A mountain comes back as a mountain, and I even demonstrated that. We were still able to sleep at that point.

But then came the awful thing, that if we substitute v2 equals infinity in this equation that TR becomes plus 2. And now we can no longer sleep, because this is absurd. All the energy that rolls in comes back, but there is something in addition that goes into that second medium.

Now, admit it. Who could not sleep that night? You should all fail this course, by the way. But in any case, I don't have to feel guilty, right? Who thought about this and said, there is something weird. I must find an explanation for that. In my case, I had to find an explanation, because I couldn't sleep.

Who found an explanation? Who could say, oh yes, don't worry about it. What was your solution?

AUDIENCE: [INAUDIBLE].

PROFESSOR: Excuse me.

AUDIENCE: Mass two is 0, so there's no energy being transmitted.

PROFESSOR: Very good. That's a very nice way of looking at it. So that's probably why you could sleep. Well actually, I'll tell you why I could sleep. But your solution is even shorter. But the reason why I want to show you what I'm going to show you is that I want to also expose you to the idea, which you already alluded to, namely that there is energy involved when we deal with pulses and when we deal with waves.

Do you remember when we have a traveling wave that the total energy per wavelength lambda-- I only did it per wavelength-- that that total energy-- perhaps you remember that-- equals 2A squared times pi squared times the tension T divided by lambda. A was the amplitude. Energy is always proportional to amplitude squared. This was the tension, and this was the wavelength.

And if v2 goes to infinity, then the wavelengths go to infinity. That's obvious, right? If something moves with the speed of light, even faster-- infinity's even faster than the speed of light-- then lambda goes to infinity, so this goes to 0. So we came to the same conclusion. Some decadent solution, TR equals plus 2, has no meaning because there's no energy in there. So I was able to sleep.

Let's now turn to normal modes of continuous media. And I suggested we go longitudinal. Because we did so much transverse stuff, let's go longitudinal. I have here a pipe, has length L, and it is open here, and it is open there. That means the overpressure or underpressure here can never build up. It's connected to the universe. So at these two boundary conditions, this p that I have there must be 0.

If I write down the general equation for a standing wave-- because normal mode solutions are standing waves-- I can write down p equals some amplitude times the sine or the cosine-- let me take the sine-- 2 pi divided by lambda times x. I'll be very general. I will introduce some phase angle alpha for which I will find the solution very shortly. And then cosine omega t or sine omega t, if you prefer that. This is a standing wave in very general terms.

Everything here is in terms-- is the space. x-- this is x. And here, all the information here deals with time, which is typical for a standing wave. I prefer always to write down for 2 pi over lambda, k, and I must observe the situation that p must be 0 at x equals 0 and also at x equals L. If p is 0 at x equals 0, immediately you see that alpha is 0.

So I'm going to rewrite it now. I'm going to write down p is p 0 times the sine of kx, So I'm going to replace this by k. I know that alpha as 0, times the cosine of omega t, and now I must meet the boundary condition that when x equals L that p is, again, 0. And that, now, breaks open a whole spectrum of possibilities in which I introduced this normal mode number n as in Nancy whereby n can be 1, 2, 3, et cetera. And then I get solutions when k of n equals n pi divided by L.

You see that immediately, because if I make x now L, then I get the sine of n pi. No matter what n is, I always get 0. So my lambda of n, which is 2 pi divided by k, is then 2L divided by n. So I can rewrite, now, this equation as p0 times the sine of n pi x divided by L. And now I have cosine omega n t, because omega n is now-- and now the frequencies which I associated with the n-th mode, n being n as in Nancy.

What, now, is the connection between this omega n and this k? Well, that connection you will find through the wave equation. You now have to substitute this result back into the wave equation, which is this one, to solve for omega. And I want to do that with you. It is not that much work.

I go to d2p dx squared. So here it is, d2p dx squared. So all I get is I get this out twice, so I get n squared pi squared over L squared-- n squared pi squared over L squared. I get a minus sign, because if I take twice the derivative, I always end up with a minus sign. But the sine comes back as a sine. And not only does the sine come back, but all the rest comes back. So I will just write p here. So this p is exactly that p. So this is the only thing that was added by taking the second derivative against x.

Now, what is d2p dt squared? Now I have to go to this function, my partial derivatives. x is constant here, but t is the one that is changing, whereas here, we had that x was changing but t was constant. So now we're going to get minus omega n squared, again, times p. The whole function comes back.

And now we are in business, because now the wave equation will tell us the connection between the two. It tells us that n squared times pi squared divided by L squared is now 1 over v squared-- that is the 1 over v squared that I have there. And there is a minus sign here, and there's a minus sign here times omega n squared. So this is not connected.

So you see the solution for omega n just is being presented to you on a silver plate. Omega n is now n pi times v divided by L. That follows from the wave equation. So if you prefer the frequency in hertz, then you have to divide this by 2 pi. So you get nv divided by 2L.

So if I want to plot now, so this is x equals 0, and this is x equals L, I can plot now here the pressure in terms of this overpressure or underpressure p, and you get a curve which looks very similar to a transverse solution. But of course, it is not transverse. It's really longitudinal. But you would get, then, that for n equals 1, you would get this mode.

There must be a pressure node here and there, because we can never build up pressure. It's connected with the universe. You can never build up overpressure. So the pressure builds up here positive, and then later in time, it will be negative, and then positive, and then negative. And when you go to n equals 2, then you get another node in pressure here. So now, you get this. Always a pressure node here. Always a pressure node there. But now you end up with another pressure node there.

And if you have any difficulties to see what the air molecules are doing, I would recommend you go back to xi space, which is the actual position of the molecules. And when you do that, you will always find that where the pressure has an anti-node, which is here, xi always has a node, and where the pressure has a node, xi always has an anti-node.

Of course, the molecules can freely move in and out. There is no problem. So the molecules can freely move in and out here. So xi has its largest amplitude is anti-nodes. That is where the pressure cannot build up. But going to xi space, actually, often helps me to see precisely what is going on with the motion of the molecules.

I have here a linear system, which is a sound cavity. It's made of magnesium. It is not air, but it is magnesium. And it is open-open on both sides. You can't have it any better. I'll make a drawing for you.

So here is my magnesium. It is a rod. It's one dimensional, and the length is 122 centimeters. And the speed of sound in magnesium is about 5,000 meters per second. When I hit that magnesium rod on the side, it wants to go into standing waves. It prefers the lowest mode. It almost always does. But it may also create second and third harmonics. So the lowest mode F1 is then v divided by 2L. So the lowest frequency F1 is v divided by 2L, which, when I calculate it, is about 250 hertz. And the actual value we measured is about 2,044.

And there may also be, when I hit it with a hammer, there may also be some that is twice as high. So that may be 4,100 hertz. That would be double the frequency. That would be the second mode.

And this is quite remarkable. So this is not filled with air, but this is filled with magnesium. But by exciting it here-- it's like blowing on the flute there-- it goes into these normal mode solutions, and it is this one that you will hear loud and clear. It's a beautiful tone. And this one, you may hear in the beginning, but the high harmonics often die out faster than the lower harmonics. So you ready for this?

Open-open. It's open on both sides. And I just bang the hell out of it. 2,044. Is that beautiful? It's oscillating like this. It's exactly oscillating the way I derived for you for air, and here, you see it that it holds [INAUDIBLE]. Only the fundamental is there. OK. I wish you luck on Thursday. I'll see you then.