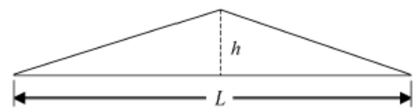
Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Problem Set #5

Problem 5.1 (French 6-12)[1] – Plucked string

A string of length L, which is clamped at both ends and has a tension T, is pulled aside a distance h at its center and released.



a) What is the energy of the subsequent oscillations?

b) How often will the shape shown in the figure reappear?

(Assume that the tension remains unchanged by the small increase of length caused by the transverse displacements.) [*Hint:* In part (a), consider the work done against the tension in giving the string its initial deformation.]

Problem 5.2 – Fourier analysis

a) Find the Fourier series of the function shown in the figure of problem 5.1.

b) If the release takes place at t=0, what will the string look like (*f(x,t)*) at time t?

c) Make sketches of the string at $t = T_1/8$, $T_1/4$ and at $T_1/2$. T_1 is the period of the lowest frequency (first harmonic). With Matlab (though not required) you can do a great job!

Problem 5.3 (French 6-14) – Fourier series

Find the Fourier series for the following functions ($0 \leq x \leq L$):

a) $y(x) = Ax(L - x)$.

b) $y(x) = A\ sin(\pi x/L)$.

c) $y(x) = \begin{cases} A\ sin(2\pi x/L) & (0 \leq x \leq L/2) \\ 0 & (L/2 \leq x \leq L) \end{cases}$

Problem 5.4 – Fourier series for a square wave

Find the Fourier series for a square wave of period 2π, and step size of 1 centered on 0 (i.e. going between -0.5 and +0.5) with the initial step positive at 0. Make a plot showing how well the series approximation matches the square wave for the first, first+second, and first+second+third non-zero terms.

Notes on coverage of this material in various textbooks

French's statement on page 231 that high-frequency waves on a string travel with lower speed than low-frequency waves, is only correct for beaded strings, not for continuous strings (like piano strings). Due to the stiffness in wires/strings, it's the other way around: the higher the frequency,

[1] The notation "French" indicates where this problem is located in one of the textbooks used for 8.03 in 2004: French, A. P. *Vibrations and Waves. The M.I.T. Introductory Physics Series.* Cambridge, MA: Massachusetts Institute of Technology, 1971. ISBN-10: 0393099369; ISBN-13: 9780393099362.

the higher the speed of propagation. Bekefi & Barrett[2] give the correct dispersion relation in strings (eq. 2.16). However, their statement that a has the dimension of length is incorrect (apart from that, a^2 in eq. 2.16 is often called α as I will do in lectures).

Before you start the remainder of this assignment, read the following interesting text which is from Crawford, *Berkeley Physics Course - Waves*[3].

Surface waves on water

At equilibrium, the surface of a body of water is flat and horizontal. When a wave is present, there are two kinds of restoring forces that tend to flatten the wave crests and restore equilibrium: one is gravity, the other is surface tension. For wavelengths of more than a few centimeters, gravity dominates. For millimeter wavelengths, surface tension dominates.

Because of the great incompressibility of water, the excess of water that appears in a wave crest must flow in from the neighboring trough regions. Individual water drops in a water wave therefore undergo a motion that is a combination of longitudinal motion (forward and backward) and transverse (up and down) motion. If the wavelength is small compared to the equilibrium depth of water, we have what are called deep-water waves. Then the individual water droplet in a traveling wave move in circles. A floating duck (or a droplet at the surface) undergoes a uniform circular motion with radius equal to the amplitude of the harmonic wave and with period equal to that of the wave. On the crest of a traveling wave, the duck has its maximum forward velocity; in a trough, it has its greatest backward velocity. Water droplets below the surface travel in smaller circles; it turns out that the radius of gyration decreases exponentially with depth. The motion is negligibly small a few wavelengths below the surface.

The dispersion relation for deep-water waves is given approximately by

$$\omega^2 = gk + \frac{S}{\rho}k^3 \tag{33}$$

where $\rho \approx 1.0$ gm/cm^3 and $S \approx 72$ dyne/cm (surface tension) for water; $g = 980$ cm/sec^2.

We shall let you show that when g and $(S/\rho)k^2$ are equal so that gravity and surface tension make equal contributions to the return force per unit displacement per unit mass (i.e., to ω^2), then the phase and group velocities are equal. You can show that this occurs at a wavelength $\lambda = 1.70$ cm. The phase and group velocities are then both 23 cm/sec. For wavelengths much less than 1.7 cm, surface tension dominates; then the group velocity is 1.5 times the phase velocity. For wavelengths much greater than 1.7 cm, gravity dominates; then the group velocity is half the phase velocity.

Table 5.1 gives wave parameters for wavelengths ranging from 1 mm (such as can be excited by a tuning fork driving a styrofoam cup full of water) up to 64 meters (very long ocean waves).

Application

Here is an example that makes use of Table 5.1. Suppose you are having a picnic at the beach.

[2]Bekefi, George, and Alan H. Barrett *Electromagnetic Vibrations, Waves, and Radiation.* Cambridge, MA: MIT Press, 1977. ISBN: 9780262520478.

[3]Crawford, Frank S. *Berkeley Physics Course. Volume 3 Waves.* New York: McGraw-Hill, 1968. ISBN-10: 0070048606; ISBN-13: 9780070048607.

Someone wonders about the wavelength of waves in the open ocean twenty or thirty miles out from the coast. You tell them to wait a minute you'll tell them the wavelength. You take out your watch and time the waves per minute, i.e., one per five seconds: $\nu = 0.2$ cps. The weather has been constant for several days, so you can assume that the waves are at steady state (aside from local winds that do not affect the big ocean swells). The frequency is thus 0.2 cps at sea, as well as at your beach. (Of course the wavelength is different, because the waves breaking on your beach are not deep-water waves. *The wavelength depends on the water depth at your local beach. The steady-state driving frequency does not.*)

Table 5.1 Deep-water waves

λ (cm)	ν (cps)	v_ϕ (cm/sec)	v_g (cm/sec)	v_g/v_ϕ
0.10	675.0	67.5	101.4	1.50
0.25	172.0	43.0	63.7	1.48
0.50	62.5	31.2	44.4	1.42
1.0	24.7	24.7	30.7	1.24
1.7	13.6	23.1	23.1	1.00
2.0	11.6	23.2	21.4	0.92
4.0	6.80	27.2	17.8	0.65
8.0	4.52	36.2	19.6	0.54
16.0	3.14	50.3	25.8	0.51
32.0	2.22	71.0	35.8	0.50
100.0	1.25	125.0	62.5	0.50
200.0	0.884	177.0	88.5	0.50
400.0	0.625	250.0	125.0	0.50
800.0	0.442	354.0	177.0	0.50
1600.0	0.313	500.0	250.0	0.50
3200.0	0.221	708.0	354.0	0.50
6400.0	0.156	1000.0	500.0	0.50

According to the table, the wavelength of the waves in the open ocean should be about 40 meters. How far have the wave crests, now breaking on your beach, traveled in the last hour? If most of the time was spent traveling in deep water, then according to Table 5.1 the phase velocity was about 8 meters/sec, i.e., about 29,000 meters per hour. Thus the waves have traveled about 30 km (20 miles) in the last hour, and since the weather has been constant for many hours you should feel confident that your estimate of wavelength in the open ocean is a good one. If you are not at the beach but are at a seismograph within ten or twenty miles of the beach, you can answer the same question.

Problem 5.5 – Phase and group velocity in your bathtub

a) Prove that for wavelengths much less than 1.7 cm, the group velocity is 1.5 times the phase velocity (use Eq. 33 above).

b) Prove that for wavelengths much greater than 1.7 cm, the group velocity is 1/2 of the phase velocity.

Home Experiment (optional)

Water wave packets. The best way to understand the difference between phase and group velocities is to make water wave packets. To make expanding circular wave packets having dominant wavelength 3 or 4 cm or longer, throw a big rock in a pond or pool. To make straight waves with wavelengths of several centimeters, float a stick across the end of a bathtub or a large pan of water. Give the stick about two swift vertical pushes with your hand. After some practice, you should see that for these packets the phase velocity is greater than the group velocity. (See Table 5.1) You will see little wavelets grow from zero at the rear end of the packet, travel through the packet, and disappear at the front. (It takes practice; the waves travel rather fast.) Another good method is to put a board at the end of a bathtub and tap the board.

To make millimeter-wavelength waves (surface tension waves), use an eye dropper full of water. Squeeze out one drop and let it fall on your pan or tub of water. First let the drop fall from a height of only a few millimeters. This gives dominant wavelengths of only a few millimeters. To see that these waves really are due to surface tension, add some soap to the water and repeat the experiment. You should notice a decrease in the group velocity when you add the soap.

Problem 5.6 – Shallow-water waves (Home experiment)

Make your own shallow-water waves for which the phase velocity $v = \sqrt{gh}$. Take a square pan a foot or two long. Fill it with water to a depth of about 1/2 or 1 cm. Give the pan a quick nudge (or lift one end and drop it suddenly). You will create two traveling wave packets, one at the near end and one at the far end, traveling in opposite directions.

a) Measure the velocity by timing one of the waves for as many pan lengths as you can (about four?). Report your findings.

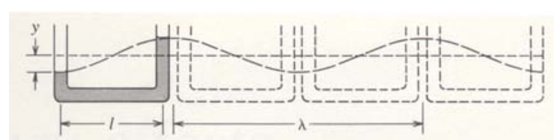
b) How well does your result agree with $v = \sqrt{gh}$?

As the depth of the water increases, you will finally get to the point where your waves change to deep-water waves.

Problem 5.7 (French 7-20) – Why are deep-water waves dispersive?

In this problem, you will derive a dispersion relation (somewhat simplified) for deep-water waves. We assume here $\lambda \gg 1$ cm so that surface tension effects can be ignored. Make sure you compare your result with that of Eq. 33 above.

Consider a U-tube of uniform cross section with two vertical arms. Let the total length of the liquid column be l. Imagine the liquid to be oscillating back and forth, so that at any instant the levels in the side are at $\pm y$ with respect to the equilibrium level, and all the liquid has the speed dy/dt.



a) Write down an expression for the potential energy plus kinetic energy of the liquid, and hence show that the period of oscillation is $\pi\sqrt{2l/g}$.

b) Imagine that a succession of such tubes can be used to define a succession of crests and troughs as in a water wave (see the diagram). Taking the result of (a), and the condition $\lambda \approx 2l$ implied

v

by this analogy, deduce that the speed of waves on water is something like $(g\lambda)^{1/2}/\pi$. (Assume that only a small fraction of the liquid is in the vertical arms of the U-tube.)

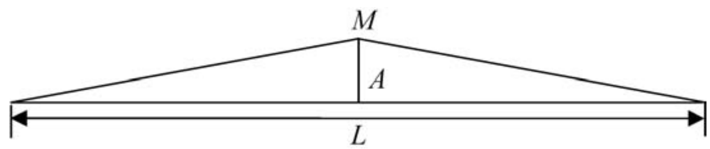
c) Use the exact result, $v = (g\lambda/2\pi)^{1/2}$, to calculate the speed of waves of wavelength 500 m in the ocean.

Problem 5.8 – Energy in waves

The energy E_λ stored in one wavelength (λ) of a vibrating string (standing wave) is: $E_\lambda = \dfrac{TA^2\pi^2}{\lambda}$ where T is the tension in the string and A is the amplitude of the anti-nodes.

a) Prove that this result is equivalent to Eq. (7-38) on page 242 in French which is: $W_{cycle} = \frac{1}{2}(\lambda\mu)\,\mu_0^2 = 2\pi^2\nu^2A^2\lambda\mu$. Note that, using $v = \sqrt{T/\mu}$ and $\nu = v\lambda$, Eq. (7-38) can be written $W_{\text{cycle}} = 2\pi^2\nu^2A^2\lambda\mu = 2\pi^2A^2\dfrac{T}{\lambda}$.

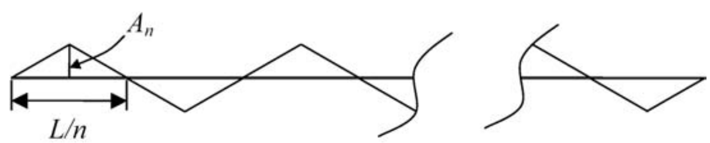
If we, for simplicity, approximate a sine-wave by triangles (see figures) then we can easily calculate how much energy it takes to give the string this triangular shape.



You pick up the string of length L in the middle M (see figure above) and move this point a distance A up. Assume that in doing so the tension T in the string does not change ($A \ll L$).

b) Calculate the amount of work that you have to do.

Now divide the same string into n equal sections, each of length L/n, as in the figure below. Move the midpoints of each of the n sections a distance A_n.



c) Calculate how much work you have to do to set up this configuration (calculate the work for one section only and multiply by n).

This simple calculation gives you the amount of energy stored in the string for given T, L, n, and A_n. If you had taken a sinusoidal shape for the string instead of a triangular shape, the energy stored would have been larger.

d) By what factor?

Notice that the energy in the n^{th} mode is proportional to n^2 (and thus proportional to ω_n^2); of course, it is also proportional to the square of the amplitude (A_n^2).

Problem 5.9 – Energy in traveling waves on a string

A string of tension T and mass per unit length μ propagates waves. Let the amplitude of the waves be A.

a) Try to reason, using the idea that a standing wave results from two traveling waves, that the kinetic energy and the potential energy in traveling waves are equal.

b) What is the kinetic and potential energy per unit length of the string?