Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Notes for Lecture #13: Deriving Electromagnetic Waves

The discovery of electromagnetic radiation (in the form of electromagnetic or EM waves) was one of the highlights of 19^{th} century physics. Mechanical and sound waves, which we have already studied, form a good basis for the study of electromagnetic waves. These include radio waves, radar (microwaves), infrared radiation, visible light, UV, X-rays, and gamma rays. In these, electric and magnetic fields are intertwined in such a way as to permit propagation, even in the "empty" space of a vacuum. The study of EM waves is based on the time-dependent Maxwell's equations (**1:30**). The versions used to study EM wave propagation are for vacuum, in which there is no charge or current density. Therefore, Maxwell's equations are written here first in general and then with all charge and current terms set to zero.

Gauss' Law, $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} = 0$, relates the divergence of the electric field to the charge density. The divergence of the magnetic field $\vec{B}$ is always zero since there are no magnetic monopoles: $\vec{\nabla} \cdot \vec{B} = 0$. Faraday's Law is $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and Ampère's Law is $\vec{\nabla} \times \vec{B} = \cancel{\mu_0 \vec{J}} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$. Note that the original form of Ampère's Law had only the first term. It was later "corrected" by the addition of the second term, the so-called "displacement current", introduced by Maxwell. It is this time derivative of the electric field which allows the electric and magnetic fields to couple to form EM waves in a vacuum. Within these equations, the "del" vector operator (**3:00**) is $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$. Applied to a scalar function ϕ, this operator produces the gradient vector $\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}$. The $\vec{\nabla}$ function can be applied to a vector in two ways. The "divergence" is found using the scalar product (dot product) resulting in the scalar: (**4:00**)

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

The "curl" of a vector function $\vec{A}$ is the vector cross product which can be written in several ways:

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

An identity for "curl of a curl" is given without proof: $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{A}$ where the final term may be written with $\vec{\nabla} \cdot \vec{\nabla} = \nabla^2$, which is called the Laplacian (**7:00**). The Laplacaian is a scalar function which when applied to a vector $\vec{A}$ gives $\nabla^2 \vec{A} = \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2}$

which has nine terms (3 derivatives of 3 components). The 3 terms in the x-component are $\left(\nabla^2\vec{A}\right)_x = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}$ so that the complete Laplacian of vector function $\vec{A}$ consists of $\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\hat{x}$ and similar terms for $\hat{y}$ and $\hat{z}$.

Taking the curl of both sides of Faradays Law gives (**9:15**) $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$ where the order of time and spatial derivatives has been interchanged on the right hand side. Applying the above identity for taking two curls would give $\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$ but the first term on the left hand side is zero due to Gauss' Law in the absence of charge, so that simply

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) \Rightarrow -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\nabla^2\vec{E}$$

We can replace the left side of the equation using Ampère's Law, $\vec{\nabla} \times \vec{B} = \epsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}$ which allows elimination of $\vec{B}$, resulting in $-\epsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = -\nabla^2\vec{E}$. Eliminating the minus signs and expanding the Laplacian gives the form (**11:10**)

$$\epsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = \frac{\partial^2\vec{E}}{\partial x^2} + \frac{\partial^2\vec{E}}{\partial y^2} + \frac{\partial^2\vec{E}}{\partial z^2}$$

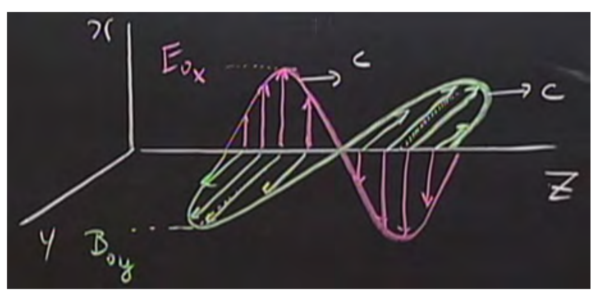
This equation was Maxwell's great victory and changed the entire outlook of science. It is a wave equation for the electric field in vacuum. It must be possible to create electric fields which move in vacuum with the speed $c = 1/\sqrt{\epsilon_0\mu_0}$. It was the genius of Maxwell in adding the displacement current term to Ampère's Law that allowed this insight. Similar reasoning through taking the curl of the curl of $\vec{B}$ brings a wave equation for the magnetic field in similar form:

$$\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

The two fields are coupled by the set of Maxwell's equations so that the wave equation for one field requires a wave equation to apply to the other (**13:00**). These are three-dimensional wave equations and there are interdependent oscillations in the electric and magnetic fields. The speed of propagation follows directly from Maxwell's equations and that "speed of light" $c = 1/\sqrt{\epsilon_0\mu_0}$ is numerically very close to 3.00×10^8 m/s. It is remarkable that the two constants which combine to give this value can be determined from entirely static experiments in electrostatics (Coulomb's Law) and magnetostatics (Ampère's Law). They are the permittivity of free space ($\epsilon_0 = 8.8 \times 10^{-12}$ in SI units) and the permeability of free space ($\mu_0 = 4\pi \times 10^{-7}$ in SI units) (**14:30**). The speed of light was well known by the time of Maxwell, having been measured in 1676 by Rœmer, who brilliantly used the eclipse times of the moons of Jupiter (these times vary due to the changing distance of Jupiter from Earth, adding an apparent delay to the timing of events) to derive the speed of light as about 2.4×10^8 m/s. The value was low since the scale of the solar system was not very accurately known at the time. In 1728, James Bradley used the technique of stellar aberration to get 3.01×10^8 m/s. In 1849, Foucault and Fizeau used rotating mirrors and disks to measure the speed of light in the laboratory as within 5% of 3×10^8 m/s. Shortly thereafter, Maxwell, knowing this measured value, could affirm that light was an electromagnetic phenomenon. In 1865, he laid the foundation of the electromagnetic theory of light.

Now we move to finding solutions to the wave equation (**16:30**). If the electric field points only along the $\hat{x}$ direction (but is defined as a traveling wave at all points in space, and propagates in the $+\hat{z}$ direction), then $E_x = E_{0_x}\cos(\omega t - kz)$ with the other components $E_y = 0$ and $E_z = 0$. In the wave equation $\epsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = \nabla^2\vec{E}$ which could have nine terms in the Laplacian, only one is nonzero, and there is only one component E_x for the time derivative to operate on, which leaves the one-dimensional wave equation (**19:00**) $\frac{\partial^2 E_x}{\partial z^2} = \mu_0\epsilon_0\frac{\partial^2 E_x}{\partial t^2}$. For Faraday's Law, the curl of $\vec{E}$ similarly has only one term if evaluated as above, so we get $\nabla\times\vec{E} = \frac{\partial E_x}{\partial z}\hat{y} = -\frac{\partial\vec{B}}{\partial t}$.

So what is $\vec{B}$? It is clear that it must be in the $\hat{y}$ direction. Doing the derivative of E_x gives $+kE_{0_x}\sin(\omega t - kz)\hat{y} = -\frac{\partial\vec{B}}{\partial t}$. Integrating in time (apart from a possible constant background magnetic field which is a constant of integration), we have $\vec{B} = \frac{kE_{0_x}}{\omega}\cos(\omega t - kz)\hat{y}$ so the one component of the magnetic field may be expressed as (**22:45**) $B_y = \frac{E_{0_x}}{c}\cos(\omega t - kz)$. The magnetic field associated with a specific electric field has a magnitude (in SI units) which is c times smaller than that of the electric field. In addition, the arguments of the cosine functions are the same, so that the fields are in phase. The magnetic field is perpendicular to the electric field, and both are perpendicular to the direction of propagation.



It is instructive to visualize a single cycle (wavelength) of an electromagnetic wave (**24:50**). Choosing a particular but arbitrary moment in time, the sinusoidal variation along the z spatial coordinate can be plotted. There is no variation in the other coordinates: the vectors shown are present at all values of x and y. More generally, the $\vec{E}$ vectors amplitude varies as a sine or cosine of z and is drawn as a smooth red curve for one cycle. Note that this is only a piece of a wave although one could well imagine that a finite pulse much as shown is possible. Individual sample vectors can be drawn at regular intervals in z and those for the electric field are parallel to the x

axis, with overall amplitude E_{0_x} (**26:00**). There must be an associated $\vec{B}$ vector in the y direction, with amplitude B_{0_y}. The entire pattern moves in the $+z$ direction with the speed of light, c.

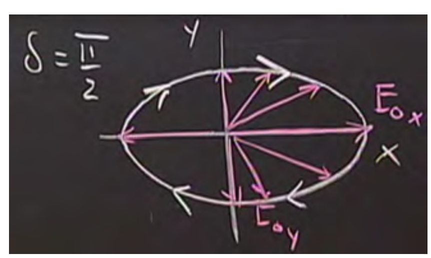
The electric and magnetic field move together: they are "married" or "stuck together." Along planes perpendicular to the direction of propagation, the values of $\vec{E}$ and $\vec{B}$ are identical at all values of x and y, since no dependence on these variables is present. This is why they are called plane waves: this may not be a completely realistic situation but it does solve the wave equations. At any point in one of these planes, the values of $\vec{E}$ and $\vec{B}$ change with time as the wave goes by, becoming positive and negative in unison, reaching their respective maximal values and zero at the same time. These equations describe traveling EM waves, standing waves will be dealt with later. Important properties of traveling EM waves include(**29:20**):

- $\vec{E}$ and $\vec{B} \perp$ direction of propagation
- $\vec{E} \perp \vec{B}$
- $\vec{E} \times \vec{B}$ is in the direction of propagation
- $\vec{E}$ and $\vec{B}$ are in phase
- $|\vec{B}| = \dfrac{|\vec{E}|}{c}$ (in SI units)

If $\vec{E}$ points only in one direction, $\hat{x}$ here, then the wave is referred to as linearly polarized. It is also possible to have an $\vec{E}$ field along the $\hat{y}$ direction, which would imply a $\vec{B}$ field along $-\hat{x}$, so that $\vec{E} \times \vec{B}$ is still in the direction of propagation ($\hat{z}$). Any linear superposition of solutions is still a solution, so it is possible to add two such solutions, and even change the phase between them:

$$\vec{E} = E_{0_x} \cos(\omega t - kz)\hat{x} + E_{0_y} \cos(\omega t - kz + \delta)\hat{y}$$

where δ is some phase angle (**32:15**). If we choose $\delta = 0$, we still get linearly polarized radiation but with a different polarization direction. When the x component of $\vec{E}$ reaches its maximum amplitude E_{0_x}, the amplitude in the y direction also reaches its maximum E_{0_y}. The total amplitude at that time is $E_{tot} = \sqrt{E_{0_x}^2 + E_{0_y}^2}$. Seeing such a wave coming toward you, you would see the $\vec{E}$ field rapidly varying back and forth at an angle in the x-y plane. It would be linearly polarized, no longer in the x or y direction but in an in-between direction (**34:00**).

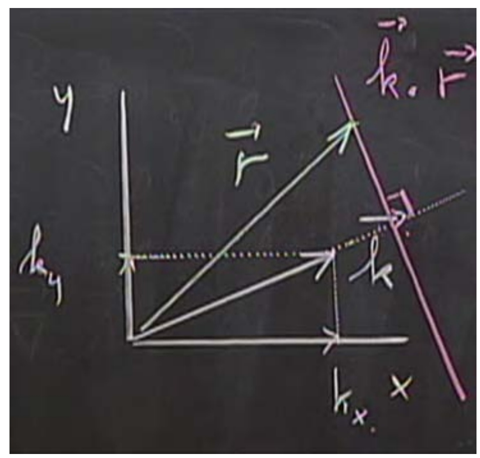


It is also possible to make the phase difference 90°, i.e. $\delta = \pi/2$, which gives rise to something interesting. When E_x has its maximum value E_{0_x}, the component in the y direction is 0 since it is 90° out of phase. After 1/4 period, E_y is maximal in amplitude (but negative), while E_x is 0. One more 1/4 period, and E_x is at its most negative while E_y is again 0, and finally after 3/4 period,

E_y is maximal and E_x again 0. There is never a time when $|\vec{E}|$ is zero, but it rotates around in an ellipse. This is called elliptically polarized radiation. This is simply a solution of Maxwell's equations with one component in the x direction and the other in the y direction, offset by 90°. Any other value of phase can also be chosen. If $E_{0_x} = E_{0_y}$, with an offset of 90° as discussed here, the path is a circle and we refer to circularly polarized radiation. If $\delta = \frac{\pi}{2}$, then it goes clockwise, and if $\delta = -\frac{\pi}{2}$, it goes counterclockwise (**36:15**).

It is straightforward to calculate the $\vec{B}$-field of any polarized wave using the rules listed previously. Simply calculate $\vec{B}$ separately for the x and y components of $\vec{E}$ and then add the results.

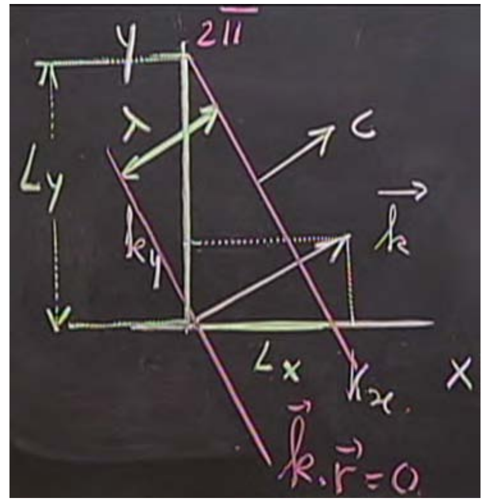
Now, consider the general three-dimensional case of a wave. With the position vector in three dimensions, $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, the general EM wave's electric field is $\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\omega t - \vec{k} \cdot \vec{r})$ where $\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ (**38:45**). The vector $\vec{k}$ is in the direction of propagation, and its magnitude is $|\vec{k}| = 2\pi/\lambda = \sqrt{k_x^2 + k_y^2 + k_z^2}$ (recall that λ is the wavelength). Now we look at the geometric meaning of $\vec{k} \cdot \vec{r}$ in the argument of the cosine. This is most easily visualized in two dimensions and then it should be clear how it extends to three dimensions. On the line perpendicular to $\vec{k}$, $\vec{k} \cdot \vec{r}$ is constant. At any point in the x-y plane $\vec{k} \cdot \vec{r} = |\vec{k}||\vec{r}| \cos\theta$ where θ is the angle between $\vec{k}$ and $\vec{r}$ (**41:30**). In this case, $r \cos\theta$ is the length of the line from the origin to the intersection of the projection of $\vec{k}$ with the perpendicular line containing $\vec{r}$. For any $\vec{r}$ which is on that line, this is true. Consider now the time $t = 0$, and with a wave propagating such that the $\vec{E}$ field along this perpendicular line is a maximum (and pointing out of the page). This is a "crest" or "mountain" in $\vec{E}$.



More specifically, consider the case that $\vec{k} \cdot \vec{r} = 4\pi$. Consider another line through the origin and parallel to the first one. On this line, $\vec{r}$ is perpendicular to $\vec{k}$ everywhere, so $\vec{k} \cdot \vec{r} = 0$ and we can also consider the perpendicular line which is between these two at $\vec{k} \cdot \vec{r} = 2\pi$. These lines, parallel to each other but perpendicular to $\vec{k}$, all show where $\vec{E}$ has a maximum pointing out of the page (**43:30**). The spacing of these lines is the wavelength λ, and the whole thing moves ahead (in the direction of $\vec{k}$) at the speed c. Extending this thinking to three dimensions, $\vec{k} \cdot \vec{r} = const.$ would form not lines, but planes, perpendicular to $\vec{k}$. In each such plane, the $\vec{E}$ vector is the same everywhere. Whether the radiation is linearly polarized, circularly polarized, or elliptically polarized is irrelevant, $\vec{E}$, and threfore $\vec{B}$, are everywhere the same in each plane, and the whole pattern moves forward at the speed of light, in the direction of $\vec{k}$. In the first case, which for simplicity was written $E_x = E_{0_x} \cos(\omega t - kz)$, $\vec{k} \cdot \vec{r}$ became simply kz. So in this example, $k_x = 0$,

$k_y = 0$ and only $k_z \neq 0$ (in fact $k_z = k$). The wavelength is then $\lambda = 2\pi/k$ in both the simple case and in three dimensions (**46:30**).

We can now look at two successive crests of an electromagnetic wave relative to the x-y plane. These are perpendicular to the vector $\vec{k}$, separated by a distance λ, and traveling in the direction of $\vec{k}$ at speed c. The distance from crest to crest along the y direction (here one of the crests happens to go through the origin) is L_y, which is much larger than λ. The analogous distance along the x axis is L_x, also larger than λ. If there is propagation in three dimensions, there may also be a similar L_z. When the wave has moved a distance λ in the $\vec{k}$ direction, the crest at the origin will now be at the position originally occupied by the second crest. However, in the same time, the crest has progressed in the y direction by a distance L_y. Since $L_y > \lambda$, the speed with which the crest moves in the y direction (v_{py}, the *phase velocity*) is bigger than c: $v_{py} = \dfrac{L_y}{\lambda}c = \dfrac{k}{k_y}c > c$ where $k^2 = k_x^2 + k_y^2 + k_z^2$. This speed of propagation of the crests along the y axis is equal to c in the case of propagation directly along y but can be far larger. (**49:25**). Although $k_y = 2\pi/L_y$ looks like the relation of a wave vector to a wavelength, it is better not to think of L_y as a wavelength, but rather that there is only one wavelength λ associated with the wave. Similarly, $v_{px} = \dfrac{k}{k_x}c$ and $v_{pz} = \dfrac{k}{k_z}c$. Consider the motion actually being along the x axis. Then the lines of constant phase would be vertical and the phase velocity along y would go to infinity. There is no violation of Einstein's theory of special relativity since no energy would flow at that speed (**51:30**).



An analogy is made with water waves traveling at speed v with a wavelength λ and hitting a shoreline at two points A and B separated by a distance L_x. The difference in arrival time for *successive* crests of the wave at either point A or B is the same, $T = \lambda/v$. It is only the difference in arrival time of the *same* wave at points A and B that depends on the angle of the waves relative to the shoreline. However, no water is moving with the (potentially much larger) speed given by this latter time difference.

Similarly, in a demonstration during the previous lecture, two aluminum plates separated by a distance a had electromagnetic radiation moving in the direction z parallel to the plates (**55:00**). It was concluded that the phase velocity component in the z direction was $v_{pz} = \omega/k_z > c$, a result that could have caused some unrest. Furthermore, if the frequency approached the cutoff, this phase velocity approached infinity. Now we know that there is, in fact, no problem since no energy flows at that very high speed for reasons similar to that for waves hitting a beach. The energy flow is at the *group* speed, which is $v_{gr} = d\omega/dk_z < c$. It is perfectly acceptable to have phase velocities greater than c, but no mass or energy can travel at that speed.

Electromagnetic waves are shown to be linearly polarized using an antenna consisting of a pair of wires driven at a frequency of 80 MHz (**57:30**). This broadcasts EM waves due to electrons accelerating along the antenna. The wavelength of such radiation is 3.75 m. A similar receiving antenna has a light bulb in the middle which lights up if current flows along it. By orienting the antennas relative to one another, a signal can be seen if the receiving antenna is parallel to the transmitting antenna, but not if they are perpendicular. Off to the side, it is necessary to turn the antenna to have it perpendicular to the direction of propagation. If one comes too close, more energy is received, to the point that the light bulb can be burned out (**1:01:00**). A similar demonstration is done using radar at 10 GHz with a 3 cm wavelength. An oscilloscope shows both the transmitted and received signals. The output is modulated with a triangular shape at about 550 Hz which is audible and contains high harmonics with an unpleasantly sharp tone. Interestingly, a person's hand absorbs such radiation. (**1:03:45**).

There are various ways to turn unpolarized light into linearly polarized light. The cheapest way is to buy a linear polarizer. These were invented by Edwin Land to change unpolarized light into 100% polarized light while reducing the intensity. An ideal polarizer reduces the light intensity by a factor of two. But what is unpolarized light? If light from, for example, a desk lamp comes straight toward you, it will contain short periods of one plane polarization followed by many others with different random orientations of polarization such that all possible angles are present. The plastic material invented by Land allows only one direction of $\vec{E}$ field polarization to emerge. When a wave with amplitude E_0 at some random angle θ comes in, only the component parallel to the preferred direction ($E_0 \cos\theta$) passes through (**1:06:45**). However, the intensity of light is proportional to the square of the electric field. The Poynting vector giving the energy transport is proportional to $\vec{E} \times \vec{B}$, but since $\vec{B}$ is, in turn, proportional to $\vec{E}$, this is proportional to E^2. For one incident direction, the intensity coming out is proportional to $\cos^2\theta$. This is known as Malus' Law. For unpolarized light, all angles appear randomly so what comes out is proportional to the average of $\cos^2\theta$ which is 1/2. So, such a polarizer turns incoming unpolarized light into 100% polarized light parallel to the preferred direction, with an intensity half (50%) that of the incoming light. In practice there is some additional absorption, so that one gets maybe 40% transmission (**1:08:30**).

Placing a first polarizer, and then a second with its preferred direction at 90^o to that of the first, is called making "crossed polarizers". Even without absorption, no light could emerge from this configuration. As a demonstration, a sheet of linear polarizer placed on an overhead projector allows through only linearly polarized light, although it is not polarized after interacting with the screen (reflecting). We cannot detect such polarization with our eyes (**1:10:00**), although some animals can, and some (such as bees) can even detect the direction. With some training, humans can detect this to some extent. A second polarizer placed in the aligned direction reduces light a bit more, due to absorption in the sheets. However, if the top one is rotated so they are crossed, all light is blocked. Strangely, placing a third linear polarizer *in between* the first two allows some light to get through (**1:103:00**). In this case, the middle polarizer allows light at an angle θ and with intensity $\cos^2\theta$ to pass through. This light then hits the second polarizer at an angle $90-\theta$ and is therefore not totally absorbed. The lecture ends with a demonstration using small polarizers given to the students. When Prof. Lewin holds up a linear polarizer in front of his face, the students can rotate their polarizers to make him appear or disappear. This effect also works in reverse, making the small polarizer in front of the students' eyes appear clear or dark to Prof. Lewin.

8.03SC Physics III: Vibrations and Waves
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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.