Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Problem Set #6 Solutions

Problem 6.1 — (Bekefi & Barrett 3.3)[1] Electromagnetic plane waves

a) First note that $\vec{B} = B_y\hat{y}$. Hence, $B_x = B_z = 0$. We now proceed by applying Maxwell's equations to $\vec{B}$ and $\vec{E}$. Gauss' Law for electricity states that

$$\vec{\nabla}\cdot\vec{E} = 0 \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \quad E_{0_x} f' k_x + E_{0_y} f' k_y + E_{0_z} f' k_z = 0$$

$$E_{0_x} k_x + E_{0_y} k_y + E_{0_z} k_z = 0 \quad \Rightarrow \vec{E}\cdot\vec{k} = 0.$$

Similarly, Gauss' Law for magnetism gives $\vec{B}\cdot\vec{k} = 0$. Ampérè's Law says that

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \quad \frac{\partial B_y}{\partial x}\hat{z} - \frac{\partial B_y}{\partial z}\hat{x} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \quad k_x B_{0_y} f'\hat{z} - k_z B_{0_y} f'\hat{x} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}.$$

Note here that $E_y = 0$ since the left side of the equation does not have a component along $\hat{y}$. Integrating the former equation with respect to t gives

$$\vec{E} = \frac{c^2 B_{0_y}}{\omega}\left(k_x f\hat{z} - k_z f\hat{x}\right) + \vec{C}(\vec{r}) = B_{0_y}\frac{c^2}{\omega} f\cdot(k_x\hat{z} - k_z\hat{x}) + \vec{C}(\vec{r}),$$

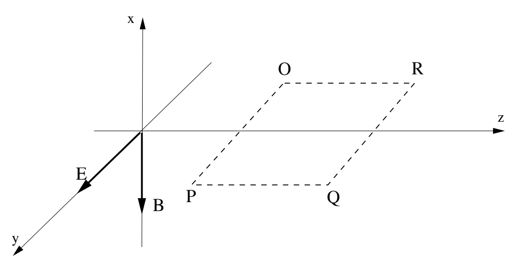
where $\vec{C}(\vec{r})$ is a constant of integration. You can quickly check that $\vec{\nabla}\cdot\vec{E} = 0$ implies $\vec{\nabla}\cdot\vec{C} = 0$. Also, we can use Faraday's Law to show that $\vec{\nabla}\times\vec{C} = 0$. The details of the algebraic steps are left as an exercise to the reader. It turns out that $\vec{\nabla}\cdot\vec{C} = 0$ and $\vec{\nabla}\times\vec{C} = 0$ imply $\vec{C} = 0$. Then, using $\omega = |k|c$, $\vec{E} = B_{0_y}\frac{c}{|k|} f\cdot(k_x\hat{z} - k_z\hat{x}) = B_{0_y} c\left(\frac{k_x}{|k|} f\hat{z} - \frac{k_z}{|k|} f\hat{x}\right)$ and so $\vec{E} = -c\,\hat{k}\times\vec{B}$. Consequently, $|\vec{E}| = c|\vec{B}|$ and $\hat{E}\times\hat{B} = \hat{k}$. Thus, $\vec{E}\perp\vec{k}$ and $\vec{E}\perp\vec{B}$. Note that the direction of propagation of the wave, $\hat{k}$, equals the direction of the Poynting vector $\vec{S}$.

b) If $k_z = 0$ then $\vec{k} = k_x$ and $\vec{E} = E_z\hat{z}$ since $\vec{k}\perp\vec{B}\perp\vec{E}$. Using the equation derived in the previous section $\vec{E} = -c\hat{k}\times\vec{B} = cB_{0_z} f\left(\vec{k}\cdot\vec{r} - \omega t + \phi\right)\vec{z}$.

Problem 6.2 — (Bekefi & Barrett 3.5) Maxwell in action

a) Since $\vec{B} = B_0\sin(\omega t - kz)\hat{x}$, $\hat{k} = \hat{z}$. Using $\vec{E} = -c\hat{k}\times\vec{B}$, $\vec{E} = -cB_0\sin(\omega t - kz)\hat{y}$.

b) The sketch shows the values of $\vec{E}$ and $\vec{B}$ at the origin and the wire loop. The EMF around the loop is:



$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S}.$$

We choose the surface of integration S to be the flat square bounded by the loop. Then, $d\vec{S}$ is normal to that surface. Since we will call positive $\mathcal{E}$ if it is in the direction $QPORQ$, then $\hat{S} = -\hat{x}$ by the right-hand rule. Furthermore, since the electromagnetic field is a plane wave, this problem has translational symmetry. Then, for convenience, we will choose the coordinate system such that

[1]The notation "Bekefi & Barrett" indicates where this problem is located in one of the textbooks used in 8.03 in 2004: Bekefi, George, and Alan H. Barrett *Electromagnetic Vibrations, Waves, and Radiation.* Cambridge, MA: MIT Press, 1977. ISBN: 9780262520478.

the origin is at point O of the loop. Then,

$$\mathcal{E} = -\frac{d}{dt}\int_0^\lambda\int_0^\lambda B_0\sin(\omega t - kz)\hat{x}\cdot(\hat{x}\,dydz) = -B_0\omega\int_0^\lambda\int_0^\lambda \cos(\omega t - kz)\,dydz.$$

Using $k = 2\pi/\lambda$, $\mathcal{E} = -\lambda B_0\omega\,\frac{\lambda}{2\pi}\sin\left(\omega t - \frac{2\pi}{\lambda}z\right)\Big|_{z=0}^{z=\lambda} = 0$. Alternatively, we can calculate the integral using the electric field of the electromagnetic wave. Notice that the path of integration is $QPORQ$.

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = \underbrace{\int_Q^P \vec{E}\cdot d\vec{l}}_{=0} + \int_P^O \vec{E}\cdot d\vec{l} + \underbrace{\int_O^R \vec{E}\cdot d\vec{l}}_{=0} + \int_R^Q \vec{E}\cdot d\vec{l} = \int_P^O \vec{E}\cdot d\vec{l} + \int_R^Q \vec{E}\cdot d\vec{l}.$$

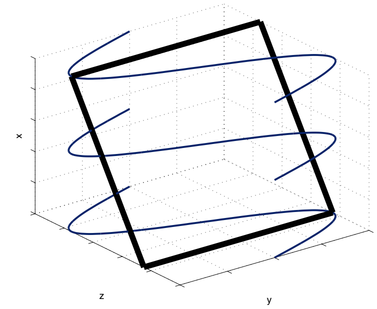
Since the loop has sides of length λ, the electric field along PO equals the electric field along RQ. Mathematically, $\vec{E}(z_{PO}) = E_0\sin(\omega t - kz_{PO})\hat{y}$ and $\vec{E}(z_{RQ}) = E_0\sin(\omega t - kz_{RQ})\hat{y}$. Since $z_{RQ} = z_{PO} + \lambda$, $\vec{E}(z_{RQ}) = E_0\sin(\omega t - kz_{PO} - 2\pi)\hat{y} = E_0\sin(\omega t - kz_{PO})\hat{y} = \vec{E}(z_{PO})$. Hence, $\int_P^O \vec{E}\cdot d\vec{l} = -\int_R^Q \vec{E}\cdot d\vec{l}$. Therefore, $\mathcal{E} = 0$ along the loop.

c) Rotating the loop about the z-axis leaves $\mathcal{E} = 0$; the argument is similar to the one in part (a). Similarly, rotating the loop about the x-axis still leaves $\mathcal{E} = 0$. However, rotating the loop about the y-axis changes the value of $\mathcal{E}$. We can calculate the inclination of the plane which maximizes $\mathcal{E}$. The EMF in the loop is defined as $\mathcal{E} = -\int_S \frac{d\vec{B}}{dt}\cdot d\vec{S}$. Hence, maximizing $|\mathcal{E}|$ implies maximizing $\left|\oint_S \frac{d\vec{B}}{dt}\cdot d\vec{S}\right|$. If the loop is flat against the YZ plane then the net flux of $\frac{\partial\vec{B}}{\partial t}$ through the plane of the loop is zero (positive flux cancels cancels an equal amount of negative flux). We wish that only positive (or only negative) flux crosses the plane of the loop. Hence, the loop must be oriented so that its projected area onto the YZ plane is half of its area. In other words, $\vec{A}\cdot\hat{x} = A/2$, where $\vec{A} = A\hat{n}$, A is the area of the loop and $\hat{n}$ is a unit vector normal to the surface of the loop. Then, $\vec{A}\cdot\hat{x} = A\cos\theta = A/2$ and $\theta = \cos^{-1}(1/2) = \pi/3 = 60°$. Hence, the EMF is

$$\begin{aligned}\mathcal{E} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{S} &= -\frac{d}{dt}\int_0^{\lambda\cos\pi/3}\int_0^\lambda B_0\sin(\omega t - kz)\,dydz \\ &= -B_0\omega\lambda\frac{\lambda}{\pi}\sin\left(\omega t - \frac{2\pi}{\lambda}z\right)\Big|_{z=0}^{z=\lambda\cos\pi/3} = 2\lambda B_0 c\cos\omega t.\end{aligned}$$

So the maximum EMF is $\mathcal{E}_0 = 2\lambda B_0 c = 2E_0$.

Alternatively, we can use Faraday's Law $\mathcal{E} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} = \oint \vec{E}\cdot d\vec{l}$. Then, maximizing $|\mathcal{E}|$ implies maximizing $|\oint_L \vec{E}\cdot d\vec{l}|$. The sketch shows the $\vec{E}$ field (blue lines) and the wire loop (thick black line) oriented at 60° with respect to the YZ plane. Note that this orientation of the loop gives the maximum $\oint_L \vec{E}\cdot d\vec{l}$.



Note that whether you calculate $-\frac{d\Phi_B}{dt}$ using only the magnetic field or $\oint \vec{E}\cdot d\vec{l}$ using only the electric field of the EM wave you will find the same result. You should not think of the electric and magnetic fields of an EM wave as being independent. Instead, you should remember that the

B-field causes the E-field and the E-field causes the B-field. They are "one and the same."

Problem 6.3: Polarized radiation

a) For angle $\alpha = \pi/4$ from the $+y$ direction $\vec{E}_{\pi/4} = \dfrac{E_0}{\sqrt{2}}\cos(\omega t - kx)(\hat{y} + \hat{z})$

$$\vec{B}_{\pi/4} = \frac{1}{\omega}k \times \vec{E} = \frac{E_0}{c\sqrt{2}}\hat{x} \times (\hat{y} + \hat{z})\cos(\omega t - kx) = \frac{E_0}{c\sqrt{2}}\cos(\omega t - kx)(\hat{z} - \hat{y})$$

For angle $\alpha = -\pi/4$ from the $+y$ direction $\vec{E}_{-\pi/4} = \dfrac{E_0}{\sqrt{2}}\cos(\omega t - kx)(\hat{y} - \hat{z})$

$$\vec{B}_{-\pi/4} = \frac{1}{\omega}k \times \vec{E} = \frac{E_0}{c\sqrt{2}}\hat{x} \times (\hat{y} - \hat{z})\cos(\omega t - kx) = \frac{E_0}{c\sqrt{2}}\cos(\omega t - kx)(\hat{y} + \hat{z})$$

b) One solution is $\vec{E} = E_0[\cos(\omega t - kx)\hat{y} + \cos(\omega t - kx + \pi/2)\hat{z}]$

$$\begin{aligned}\vec{B} &= \frac{1}{\omega}k \times \vec{E} = \frac{E_0}{c}\hat{x} \times [\cos(\omega t - kx)\hat{y} + \cos(\omega t - kx + \pi/2)\hat{z}] \\ &= \frac{E_0}{c}[\cos(\omega t - kx)\hat{z} - \cos(\omega t - kx + \pi/2)\hat{y}]\end{aligned}$$

This is usually called left-handed circular polarization, but is called right-handed by Bekefi and Barrett. The second solution is $\vec{E} = E_0[\cos(\omega t - kx)\hat{y} + \cos(\omega t - kx - \pi/2)\hat{z}]$

$$\begin{aligned}\vec{B} &= \frac{1}{\omega}k \times \vec{E} = \frac{E_0}{c}\hat{x} \times [\cos(\omega t - kx)\hat{y} + \cos(\omega t - kx - \pi/2)\hat{z}] \\ &= \frac{E_0}{c}[\cos(\omega t - kx)\hat{z} - \cos(\omega t - kx - \pi/2)\hat{y}]\end{aligned}$$

This is usually called right-handed, but is called left-handed by Bekefi and Barrett.

Problem 6.4 — Linear polarizers – Malus' law + absorption

The amplitude of the E-field of an EM wave transmitted through a linear polarizer is $E_T = E\cos\theta$, where E_T and E are the E-field amplitudes of the transmitted and incident waves, respectively, and θ is the angle between the polarization of the incident wave and the direction of polarization of the polarizer. Thus, the intensity is reduced by $\cos^2\theta$. Since $\langle\cos^2\theta\rangle = 1/2$, half of unpolarized light passes through a perfect polarizer, hence the designation HN50.

Furthermore, since the polarizers are HN30, the transmitted intensity through one polarizer is $I = (0.5 \times 0.7)I_u$, where I_u is the intensity of the unpolarized light. This I is I_0 in our problem. The intensity through two polarizers is $I = I_0(0.7\cos^2\theta_{12})$, where θ_{12} is the angle between the polarization axes of the first and second polarizers. Similarly, the intensity through three polarizers is $I = I_0(0.7\cos^2\theta_{12})(0.7\cos^2\theta_{23})$, where θ_{23} is the angle between the polarization axes of the second and the third polarizers. Let's examine each case individually:

F: The unpolarized light passes through only one polarizer, so $I = (0.5 \times 0.7)I_u = I_0$.

G: The light passes through two polarizers at right angles, so $I = I_0(0.7\cos^2\pi/2) = 0$.

H: Two polarizers: $\theta_{12} = \pi/6$. Hence, $I = I_0(0.7\cos^2\pi/6) = 0.525I_0$.

K: Three polarizers: $\theta_{12} = \pi/6$ and $\theta_{23} = \pi/3$. $I = I_0(0.7\cos^2\pi/6)(0.7\cos^2\pi/3) \approx 0.368I_0$.

L: Note that this case is physically identical to H so $I = 0.525I_0$.

M: Two polarizers: $\theta_{12} = \pi/3$. Hence, $I = I_0(0.7\cos^2\pi/3) = 1.4I_0$.

N: The light passes through only one polarizer, so $I = I_0$.

Problem 6.5: (Bekefi & Barrett 4-1) Radiation from an accelerated charge

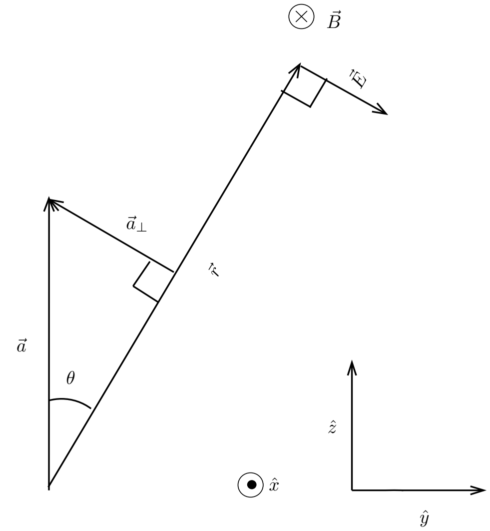
$\vec{a}_\perp$ is the component of the acceleration of the charge in a direction perpendicular to the position vector of the observer. θ is the angle between the direction of acceleration and the position vector of the observer. $t' = t - \frac{r}{c}$

$$\vec{E}(\vec{r},t) = \frac{-q\vec{a}_\perp(t')}{4\pi\epsilon_0 rc^2}\ \text{Vm}^{-1} \quad \vec{B}(\vec{r},t) = \frac{\hat{r}}{c} \times E(\vec{r},t)\ \text{Wm}^{-2}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}\ \text{Wm}^{-2}$$

$$E(\vec{r},t) = \frac{-qa(t')\sin\theta}{4\pi\epsilon_0 rc^2} \quad |\vec{S}(\vec{r},t)| = \frac{q^2a^2(t')\sin^2\theta}{16\pi^2\epsilon_0 r^2c^3}$$

$$P(t) = \int_0^\pi |\vec{S}(\vec{r},t)| 2\pi r^2 \sin\theta d\theta = \frac{q^2a^2(t')}{6\pi\epsilon_0 c^3}\ \text{Watt}$$



a) Arrival time at all the three observers A, B and C is $t_{arrival} = R/c$. The direction of the electric field at the point of observation is anti-parallel to the component of the acceleration perpendicular to the position vector. The direction of the magnetic field is the cross product of the position vector with the electric field. The figure below shows the directions to the 3 observers.

- Observer A

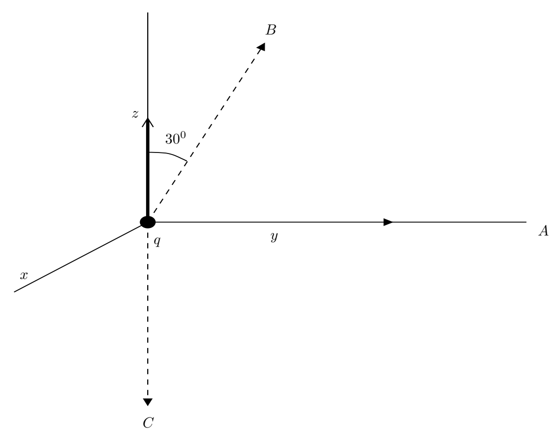
$$\vec{E}_A = \frac{qa(t')}{4\pi\epsilon_0 Rc^2}\sin\theta_A(\vec{r}_A \times \hat{x}) \quad \theta_A = \frac{\pi}{2}$$
$$= \frac{-qa(t')}{4\pi\epsilon_0 Rc^2}\ \hat{z}$$

- Observer B

$$\vec{E}_B = \frac{qa(t')}{4\pi\epsilon_0 Rc^2}\sin\theta_B(\vec{r}_B \times \hat{x}) \quad \theta_B = \frac{\pi}{6}$$
$$= \frac{1}{2}\frac{qa(t')}{4\pi\epsilon_0 Rc^2}\left(\frac{\sqrt{3}}{2}\hat{y} - \frac{1}{2}\hat{z}\right)$$

- Observer C

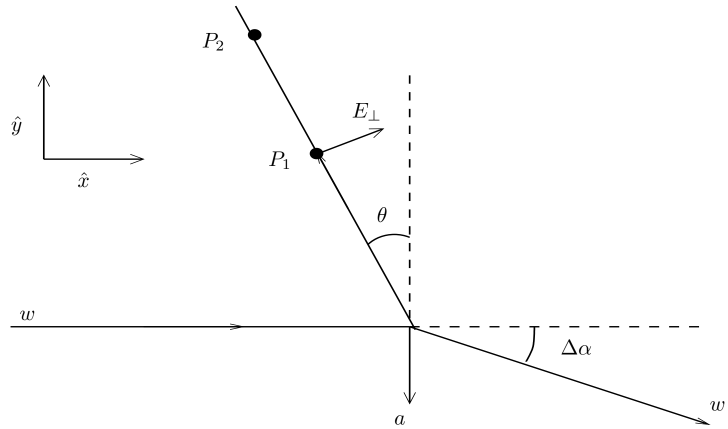
$$\vec{E}_C = \frac{qa(t')}{4\pi\epsilon_0 Rc^2}\sin\theta_C(\vec{r}_C \times \hat{x}) \quad \theta_C = 0$$
$$= 0$$



b) As $|B| = |E|/c$, hence the relative strengths of the magnetic field B are the same as the relative strengths of the electric field E in Part(a) at the three observation points. The arrival time of the magnetic field at the three observers A, B and C is $t_{arrival} = R/c$. The direction of the induced magnetic field at the three points is in the $-\hat{x}$ direction.

Problem 6.6: (Bekefi & Barrett 4-2) Radiation from an accelerated charge

a) A point charge $+q$ from time interval $t = t_0$ to $t = t_0 + \Delta t$ feels a force perpendicular to its trajectory, and moves along a new trajectory without changing its speed $|\vec{w}|$. Since the angle $\Delta\alpha$ is small, the acceleration along the x axis is negligible and does not effect the answer. The only significant acceleration of the point charge is along the $-y$ direction.



$$\vec{a} = \frac{\Delta V_y}{\Delta t}\hat{y} = \frac{w\sin\Delta\alpha}{\Delta t}\hat{y} \simeq w\frac{\Delta\alpha}{\Delta t}\hat{y}$$

$$a_\perp = a_y \sin\theta = w\frac{\Delta\alpha}{\Delta t}\sin\theta$$

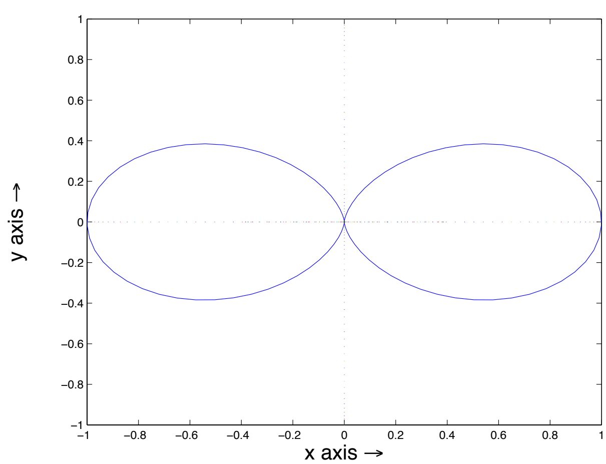
where $a_\perp$ is the component of acceleration perpendicular to the position vector of the distant point P_1. Then the electric field at point P_1 is anti-parallel to $a_\perp$ and is oriented as shown in the figure.

$$E = \frac{q}{4\pi\epsilon_0 r}\frac{a_\perp}{c^2}(\hat{r}_{P1}\times\hat{z}) = \frac{q}{4\pi\epsilon_0 r}\frac{v}{c^2}\frac{\Delta\alpha}{\Delta t}\sin\theta(\cos\theta\hat{x} + \sin\theta\hat{y})$$

So at a distant point P_1 the electric field caused by the acceleration has the direction $(\cos\theta\hat{x}+\sin\theta\hat{y})$ where θ is the angle shown in the figure.

b) The radiation intensity $\propto |\vec{E}_\perp|^2 \propto \sin^2\theta$. So it is most intense in the $x - z$ plane.

c) The least intense direction is along the y axis. The figure shows the radial plot of variation of intensity with angle θ from along the $+y$ direction.



d) $\vec{B}(\vec{r},t) = \hat{r}\times\dfrac{\vec{E}(\vec{r},t)}{c}$

$\Rightarrow \quad \vec{B} \simeq \dfrac{E_\perp}{c} \propto \dfrac{1}{r}$ so, the amplitude decreases by a factor of 2.

e)

$$\Delta E_{radiated} = P\Delta t = \frac{q^2 a^2 \Delta t}{6\pi\epsilon_0 c^3} = \frac{q^2 w^2}{6\pi\epsilon_0 c^3}\left(\frac{\Delta\alpha}{\Delta t}\right)^2 \Delta t$$

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