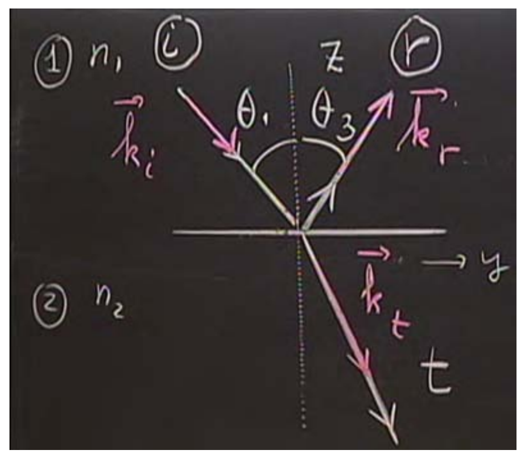
Massachusetts Institute of Technology OpenCourseWare

8.03SC **Fall 2012**

Notes for Lecture #18: Interactions of Light with Nonconductors

The boundary conditions for dielectrics are very different from those for ideal conductors. Electric fields and changing magnetic fields *are* possible inside dielectrics. Maxwell's equations still apply provided we replace ϵ_0 with $\kappa_e\epsilon_0$, and μ_0 with $\kappa_m\mu_0$, where κ_e is the dielectric constant, and κ_m is the magnetic permeability. Except in ferromagnetic materials, κ_m is always very close to 1.0. Where the speed of light in vacuum was $c = 1/\sqrt{\epsilon_0\mu_0}$, these substitutions give a speed of propagation of $v = c/\sqrt{\kappa_e\kappa_m} = c/n$ where n is the *index of refraction* characteristic of the dielectric (**2:35**). The dielectric constant is a strong function of frequency, which is the cause of dispersive materials. For light at 5×10^{14} Hz in water, $\kappa_e \approx 1.77$, making $n = 1.33$; for glass typically $n = 1.5$. Both vary with frequency, and there are many types of glass with different indices.

The boundary conditions for light passing from one dielectric medium to another are expected to result in both a reflected and transmitted (refracted) beam. Assume that a boundary at z=0 which is horizontal (denoted the y direction) separates medium "1" above from medium "2" below, with indices of refraction of n_1 and n_2, respectively. The normal to the boundary will be used as a reference from which to measure angles. The incident beam has wave vector $\vec{k}_i$, the reflected beam has $\vec{k}_r$, and the transmitted beam (in medium "2") has $\vec{k}_t$ (**5:45**).



The angle of the incident beam from the normal is θ_1 and that of the reflected beam θ_3. Soon we will show that $\theta_1 = \theta_3$. The frequency is the same for all beams and $\omega = kv$. Since v is the same for the incident and reflected beams in medium "1", the magnitudes of $\vec{k}_i$ and $\vec{k}_r$ are the same, whereas $\vec{k}_t$ in medium "2" will not be the same. However, $\omega = k_1 v_1 = k_2 v_2$. Using the speeds in the two media, we can write $k_i(c/n_1) = k_t(c/n_2)$ or $k_i/n_1 = k_t/n_2$. The incident reflected and transmitted waves are written using exponential notation, as:

$$\vec{E}_i = \vec{E}_{0_i} e^{j(\omega t - \vec{k}_i \cdot \vec{r})} \qquad \vec{E}_r = \vec{E}_{0_r} e^{j(\omega t - \vec{k}_r \cdot \vec{r})} \qquad \text{and} \qquad \vec{E}_t = \vec{E}_{0_t} e^{j(\omega t - \vec{k}_t \cdot \vec{r})}, \text{ respectively.}$$

At the interface, the incident and reflected waves must add up to match the refracted wave. Satisfying this requirement at all times forces the three frequencies to be the same. In addition, the three waves must be either in phase, or out of phase by 180°. For any given time the relative phase is determined by the geometric terms. At $z = 0$ we must have $\vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} = \vec{k}_t \cdot \vec{r}$ (**8:40**).

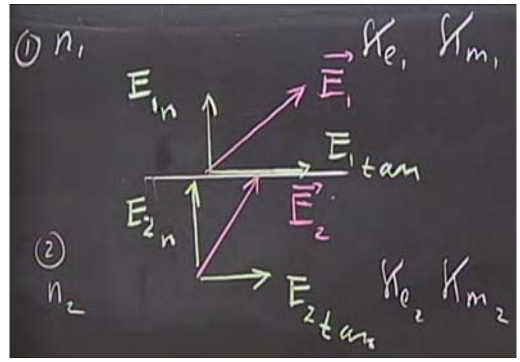
We can consider the plane in which all the vectors lie to be $x = 0$. Then at the point of intersection of the k vectors, $\vec{r} = (0, y, 0)$, and the dot products can be explicitly done. The angles between the vectors are 90° ($\pi/2$) minus the angles to the normal, and $\cos(\pi/2 - \theta) = -\sin\theta$, so we get $k_i y \sin\theta = k_r y \sin\theta_3 = k_t y \sin\theta_2$. Since magnitudes k_i and k_r are the same, the first two give $\theta_1 = \theta_3$. The first and last give $k_i y \sin\theta_1 = k_t y \sin\theta_2$, and with $k_i/n_1 = k_t/n_2$, we can rearrange to get $k_i \sin\theta_1 = k_t \sin\theta_2 = (n_2/n_1) k_i \sin\theta_2$, i.e. $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

The first is the law of reflection that the angle from the normal is the same exiting as entering. The second, known as *Snell's Law*, relates the angles from the normal with change in refractive index, i.e. propagation speed. Snell's law can be obtained from only this consideration and applies to any waves when they go between media where the speeds differ. It is not directly a consequence of Maxwell's equations. It holds, for example, for waves in water or sound waves, when they pass between media where the speeds differ. Willibord Snellius, after whom the law is named, was a Dutch mathematician who discovered this relationship empirically in 1621, before it was known that the speed of light is different in different media (**10:10**).

As an example of Snell's law, consider light going from air ($n_1 = 1$ to a very good approximation) to water, $n_2 = 1.33$. The equations relating the angles of incidence and refraction are not too exciting until an incidence angle of 90°, i.e. basically parallel to the water surface. In this case, the angle inside the water is considerably less, $\theta_{2_{max}} = 48.7°$. A more interesting case is that of going from water to air. In this case, $n_1 = 1.33$ and $n_2 = 1$. If $\theta_1 > 48.7°$, then it is not possible to solve for the refracted angle since $\sin\theta_2 > 1$. Nature resolves this problem by having *no* refracted light. This is *total internal reflection* with all of the incident light reflected at the angle θ_1. The incident angle for this to happen (48.7° between water and air) is called the *critical angle*, denoted θ_{cr}. In the general case $\sin\theta_{cr} = n_2/n_1$. There is only a critical angle if $n_1 > n_2$, i.e. in going from an "optically dense" medium to an "optically less dense" medium (**14:45**). Lasers and a transparent water tank are used to demonstrate these phenomena.

In fiber optics, total internal reflection is important since it allows transmission through glass fibers with no loss in intensity, even after many kilometers of fiber (and many internal reflections) (**17:45**). A demonstration is done with a cable consisting of 5000 optical fibers, each $1/20^{\text{th}}$ of a mm in diameter. The cable can be bent and even knotted and the outgoing intensity does not change. The many fibers can directly transmit an image if the fibers come out at the far end in the same arrangement as they are at the input side (**21:45**).

Now consider the intensity of the three beams at a dielectric interface. The procedure uses Maxwell's equations and is similar to what was done for conductors. Again, the two media are separated by a horizontal boundary, with the upper one having an index of refraction n_1, dielectric constant κ_{e_1} magnetic permeability κ_{m_1}, and the lower one having n_2, κ_{e_2}, κ_{m_2}. As before, we decompose the wave above the interface, $\vec{E}_1$, into tangential and normal components, $E_{1_{tan}}$ and E_{1_n}. The refracted wave can be similarly decomposed into $E_{2_{tan}}$ and E_{2_n}. As for conductors, the Maxwell's equations with "divs" will lead to pillboxes and the ones with "curls" will lead to closed loops (**24:50**). Previously, Prof. Lewin showed two out of four, leaving two as an exercise. Now, he will do just one out of four, namely Faraday's Law, $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, or in integral form $\displaystyle\oint_{\text{closed loop}} \vec{E} \cdot d\vec{l} = -\dfrac{\partial \phi_B}{\partial t}$, where the magnetic flux ϕ_B through any open surface is related to the integral of E around the attached closed loop.

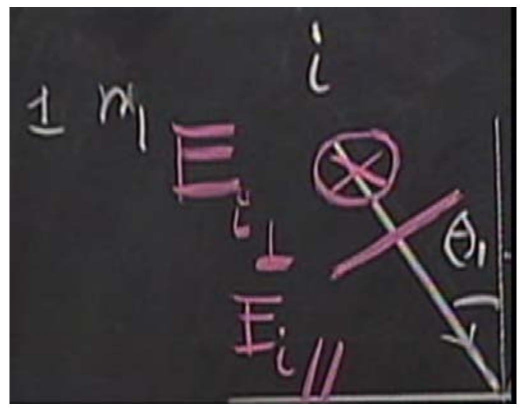


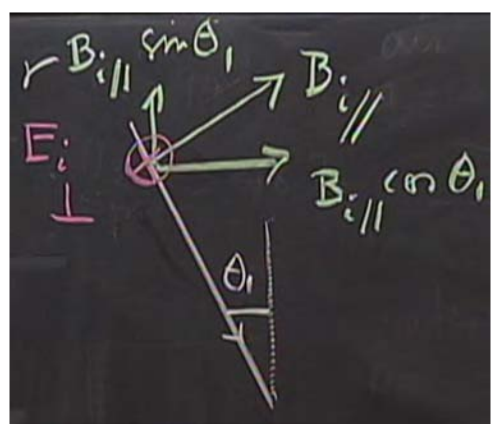
The chosen loop is a path perpendicular to the interface with total length dz in that direction, and length L parallel to the interface. Contributions from the ends will disappear when we take the limit $dz \to 0$. Along the top we get a contribution of $E_{1_{tan}} L$ but on the bottom the path takes us in the opposite direction so that contribution is $-E_{2_{tan}} L$, giving $E_{1_{tan}} L - E_{2_{tan}} L = -\dfrac{\partial \phi_B}{\partial t}$. However, as $dz \to 0$, the area over which we integrate the (finite) B field goes to zero, and thus so does the flux and its time derivative. The final result and first boundary condition is $E_{1_{tan}} - E_{2_{tan}} = 0$, or simply $E_{1_{tan}} = E_{2_{tan}}$ (**29:00**). This is not that different from the condition $E_{tan} = 0$ for a conductor, since that is true both just above the surface and just below it.

Derivation of the other three boundary conditions using Maxwell's equations is not difficult and the details are left as an exercise. The normal component is related to surface charge in a way that superficially resembles that for conductors, but there is a huge difference since, in insulators, charges cannot move around. This boundary condition can be written $\kappa_{e_1} E_{1_n} - \kappa_{e_2} E_{2_n} = \rho/\epsilon_0$. which is analogous to $E_n = \rho/\epsilon_0$ for conductors. With electromagnetic waves, in the latter case the surface charge density will vary enormously but, in the case of insulators, there is no change in the surface charge density due to the wave: the surface charge is static. In many cases it would be zero. The final two boundary conditions are for magnetic fields and we have $B_{1_n} = B_{2_n}$ (continuity of normal components) and $\dfrac{B_{1_{tan}}}{\kappa_{m_1}} = \dfrac{B_{2_{tan}}}{\kappa_{m_2}}$ for tangential components (**32:00**).

	Dielectrics (Ideal Insulators)	Ideal Conductors
	$\kappa_{e_1} E_{1_n} - \kappa_{e_2} E_{2_n} = \rho_s/\epsilon_0$	$E_n = \rho_s/\epsilon_0$
Summary Table of Boundary Conditions	$E_{1_{tan}} = E_{2_{tan}}$	$E_{tan} = 0$
	$B_{1_n} = B_{2_n}$	$B_n = 0$
	$B_{1_{tan}}/\kappa_{m_1} = B_{2_{tan}}/\kappa_{m_2}$	$\lvert B_{tan}\rvert = \mu_0 \lvert J_s\rvert$

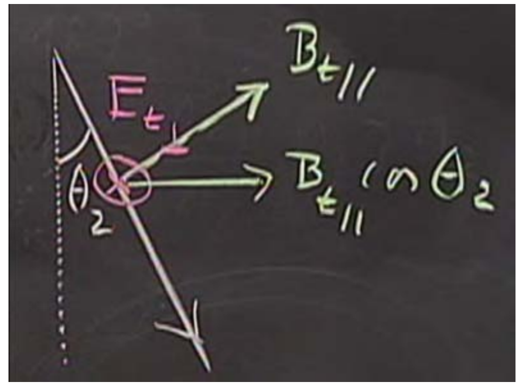
A more complicated topic is finding the intensities of transmitted and reflected light relative to the incident light. This leads to four equations which are known as *Fresnel's Equations*. Two of the four will be derived, leaving the other two as an exercise. Consider the same transition as before from n_1 to n_2. The reflected angle from the normal is the same as the incident angle, denoted θ_1. Transmitted radiation will exit at an angle θ_2. As in all EM waves, the E vector in the incident radiation must be perpendicular to the direction of propagation, i.e. in a plane perpendicular to the blackboard (the blackboard contains all of the k vectors). This means that the incident radiation can have its E field decomposed into parts which, while lying in this plane perpendicular to the incident k vector, also lie perpendicular to the blackboard ($E_{i_\perp}$) or parallel to it ($E_{i_{||}}$). It does not matter what the plane of polarization is, it could even be time varying as in circularly polarized light (**35:45**). We will proceed in detail for the perpendicular component, predicting what part of it shows up in the reflected and in the transmitted beams. The parallel component is left as an exercise. The incident perpendicular E field component must have an associated B field vector, with $\vec{E} \times \vec{B}$ in the direction of propagation. That constrains $\vec{B}$ to be in the same plane perpendicular $\vec{k}_i$ that $\vec{E}$ is in. Further, Faraday's Law that in this case of $E_{i_\perp}$, the associated B must be parallel to the blackboard, do to solve the $E_{i_\perp}$ case, we have only an associated $B_{i_{||}}$. If the angle of incidence is θ_1, we can decompose this $B_{i_{||}}$ further into its parts parallel and perpendicular to the interface, respectively $B_{i_{||}} \cos\theta_1$ and $B_{i_{||}} \sin\theta_1$. This decomposition allows us to use the tangential boundary conditions $B_{1_{tan}}/\kappa_{m_1} = B_{2_{tan}}/\kappa_{m_2}$. A similar thing is done for the E field boundary conditions.





Now consider the reflected wave (**39:00**). An incoming $E_{i_\perp}$ is expected to produce an outgoing $E_{r_\perp}$, although this might not be true if the medium is nonlinear in some way. Recall that this wave is at the same angle θ_1 as the incident wave. The associated B component is in the plane of incidence (the blackboard) and is called here $B_{r_{||}}$. The reflected B component can be decomposed and for the tangential boundary condition, we only need the part parallel to the interface, $B_{r_{||}} \cos\theta_1$.

Now consider the second medium, where the angle from the normal is θ_2. The transmitted wave arising from an incident perpendicular E field would be expected to have only a perpendicular component, $E_{t_\perp}$. As before, the associated B must be in the plane of the blackboard, $B_{t_\perp}$, and the component needed to apply the tangential boundary condition is $B_{t_\perp} \cos\theta_2$. In order to understand all of these vectors and components, watching this section of the video is *absolutely critical* (**42:00**)!



Recall that in dielectrics, the ratio of the magnitudes of E and B is $|B_{||}| = \frac{E_\perp}{v} = \frac{E_\perp}{c} n$, applicable in either medium. When appling the tangential boundary condition, the incident and reflected E fields superpose. So, $E_{i_\perp} + E_{r_\perp} = E_{t_\perp}$. Similarly for B, but with the slight complication of the projection angle, the tangential boundary condition (using $\kappa_m \approx 1$ for most dielectrics) gives $B_{i_{||}} \cos\theta_1 - B_{r_{||}} \cos\theta_1 = B_{t_{||}} \cos\theta_2$ (**45:00**). The ratios of E to B are used to eliminate B, giving $E_{i_\perp} n_1 \cos\theta_1 - E_{r_\perp} n_1 \cos\theta_1 = E_{t_\perp} n_2 \cos\theta_2$. These two equations for the $E_\perp$ terms (using Snell's Law to give the angles) have three unknowns, but only the ratios of the electric fields, $(E_t/E_i)_\perp$ and $(E_r/E_i)_\perp$ are meaningful. The solutions, with subscript 0 denoting the amplitude of E, and r and t as a shorthand for ratios of reflected and transmitted to incident (respectively) are (**48:00**):

Fresnel equations:

$$r_{||} = \frac{E_{0r_{||}}}{E_{0i_{||}}} = \frac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1} = -\frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}$$

$$r_\perp = \frac{E_{0r_\perp}}{E_{0i_\perp}} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} = -\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}$$

$$t_{||} = \frac{E_{0t_{||}}}{E_{0i_{||}}} = \frac{2 n_1 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1} = \frac{2 \sin\theta_2 \cos\theta_1}{\sin(\theta_1 + \theta_2)\cos(\theta_1 - \theta_2)}$$

$$t_\perp = \frac{E_{0t_\perp}}{E_{0i_\perp}} = \frac{2 n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_1} = \frac{2 \sin\theta_2 \cos\theta_1}{\sin(\theta_1 + \theta_2)}$$

Starting off with the indices of refraction and the incident angle θ_1, Snell's Law gives θ_2 and the Fresnel equations can be used to get the E field ratios for the two directions of polarization. (**52:00**) In the case that $\theta_1 = \theta_2 = 0$, the incident beam comes along the normal line, so this is called *normal* *incidence*. In this case there is really no difference between parallel and perpendicular components, which become identical $\frac{E_r}{E_i} = \frac{n_1 - n_2}{n_1 + n_2}$ and $\frac{E_t}{E_i} = \frac{2n_1}{n_1 + n_2}$.

As a practical example, consider a transition from air to glass so that $n_1 = 1.0$ and $n_2 = 1.5$, giving $E_r/E_i = -0.2$. The minus sign means that a 180° phase change takes place. Furthermore, $E_r/E_i = +0.8$ with the plus sign showing that there is no phase change. As for a string, the reflected wave can change phase but the transmitted one never can. A plot illustrates the four Fresnel equations for the case of air to glass. The ratios, in the range -1 to 1, are shown for angles of incidence from 0° (normal incidence) to 90° (called grazing incidence) (**56:00**).

Light intensity can be found using the Poynting vector, which is proportional to E^2/v. The incident and reflected waves are in the same medium so the v cancels out and $\frac{I_r}{I_i} = \left(\frac{E_r}{E_i}\right)^2$. For $\theta_1 = 0°$, this intensity ratio is 0.04, a number well known to astronomers. It means that when light strikes a glass surface (a lens, for example) at normal incidence, 4% is reflected. To conserve energy, the rest of the light must go through, i.e. $I_t/I_i = 0.96$. Finding this is left as an exercise, in which it is very important to take into account the velocity in medium 2.

If one looks at the first equation, a strange thing occurs if $\theta_1 + \theta_2 = 90°$. The tangent of 90° being infinity, $r_{||} \to 0$, and there is an incident angle for which *none* of the light is reflected. This condition does not apply to the perpendicular component, however: some of it is still reflected at this angle. However, since none of the parallel component is reflected, the reflected light is 100% polarized in the direction perpendicular to the plane of incidence. The unique angle at which this happens is called the *Brewster angle*, denoted here θ_{Br} (**1:00:05**).

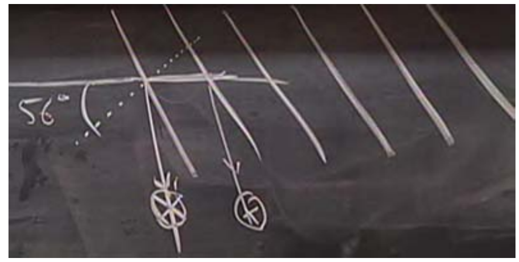
If $\theta_1 + \theta_2 = 90°$, then $\cos\theta_1 = \sin\theta_2$. Snell's Law gives $n_1 \sin\theta_1 = n_2 \sin\theta_2$ so $n_1 \sin\theta_1 = n_2 \cos\theta_1$, and therefore $\tan\theta_1 = \tan_{Br} = n_2/n_1$. Going from air to glass of index 1.5, $\theta_B r = 56.3°$. By Snell's law, it can be found that $\theta_2 = 33.7°$ (giving the expected sum of 90°). At this angle $r_{||} = 0$, but it is necessary to use the Fresnel equations to find that $r_\perp = -0.385$, so about $(0.385)^2$, or 0.15, i.e. 15%, of perpendicularly polarized light is reflected. If unpolarized light comes in, which can be broken down as 50% parallel and 50% perpendicular, then 7.5% of it will be reflected with 100% perpendicular polarization. There is also a Brewster angle for going from glass to air, 33.7° (**1:05:15**). Similarly, the parallel polarized light goes through while the perpendicular polarized light is partially reflected, but 100% polarized. If one increases the angle further in this case, the critical angle is reached at which there is total internal reflection. This is possible only in going from a denser to a less dense medium, such as in this case going from glass to air.

The Fresnel equations are very powerful, allowing calculation of intensities as a function of the refractive indices and incident angle. If the light is polarized in some arbitrary manner, one decomposes it into a parallel and perpendicular component with respect to the plane of incidence. One "turns the crank" on these components and gets the reflection and transmission for each. This can be done for any type of light, even if the polarization vector changes with time as in elliptically or circularly polarized light. Recombining, one can calculate to what degree the processed light is polarized. The degree of *linear* polarization can be defined as $V = \left| \frac{I_{||} - I_{\perp}}{I_{||} + I_{\perp}} \right|$ which uses the difference in intensities in the directions of polarization over their sum. If this ratio is zero, the light is totally unpolarized, and if it is one, it is 100% linearly polarized, and in between indicates partially polarized light. Rotating a linear polarizer would gives differences in intensity (**1:08:10**).

What should be the direction of polarization for polarized sunglasses? One has to think about the reason one would want such sunglasses. If light reflects from a dielectric surface, such as water, it will be at least partially polarized perpendicular to the plane of reflection, i.e. horizontally for many surfaces. In that case a choice of vertical polarizer will greatly reduce this reflected (undesired glare) light. A demonstration uses a large sheet of polarizing material which is rotated in a beam of reflected light at near the Brewster angle. When the polarizer is horizontal, light passes through to light up Prof. Lewins face but when the sheet is vertical, it cuts out all of this light, and his face becomes darkened (**1:12:15**). Looking into the beam with a TV camera, when the sheet is horizontal, the glare from a picture covered with glass obscures the picture, while if it is vertical this glare component is removed and one can see the picture (which scatters light essentially diffusely and does not affect polarization). This is basically what polarizing sunglasses do, select vertically polarized light, which is not present in reflective glare. One can check this for oneself by using a polarizer to look at reflections from metal (which does not polarize) and contrasting with reflections from dielectric materials like plastic, varnished wood, or leather.

The ultimate demonstration of the Brewster angle is to amplify the polarizing effect of reflection from glass (**1:15:15**). About 7.5% of unpolarized light is reflected at the Brewster angle, and that reflected light is 100% polarized perpendicular to the plane of incidence. In the demo, the plane of incidence is horizontal, so the reflected light is polarized vertically. Before this demo, however, think about what would happen if more panes of glass were put into the beam. The parallel component goes through the glass 100%: however, bear in mind that since only 15% of the perpendicular component is reflected, 85% goes through. There is still a significant perpendicular component left to reflect from another sheet of glass at the Brewster angle.

If enough sheets of glass were set up one after the other, basically all of the perpendicular polarized light would go out the side, and the transmitted beam would be 100% parallel polarized (**1:17:30**). For the demo, a stack of several hundred glass plates is used. To the side we expect to see 100% vertically polarized light either from a single sheet of glass or from the stack. The single sheet will pass through light missing only 7.5% of its original intensity and not very polarized. However, the stack will have, by multiple extractions of the vertically polarized light, allowed only the horizontally polarized light to pass and be essentially 100% polarized in that direction.



The parallel plane in this demo is horizontal, and the degree of polarization in either the reflected or transmitted beams can be shown by rotating a polarizing sheet. A projector produces a strong round beam of white, unpolarized light. By entering at the Brewster angle of about 56°, the light reflects a total of 112° and ends up on a screen a bit to the left of the glass plate. Reflection on the screen disrupts the polarization, so the large polarizer must be inserted into the beam to determine the polarization: observing the screen does not show it well. Far to the left, in the direct line of the beam, about 93% of the light goes through to make a large round spot, while about 7.5% was reflected into a small spot off to the side.

Initially, a single pane of glass is set near but not at the Brewster angle. It is shown that the reflected light is highly, but not 100%, polarized, since some gets through for all angles of the polarizer. When the angle is closer to the Brewster angle, blocking vertical polarization removes 100% of the reflected light. The transmitted light has only 15% of the vertically polarized light removed, so is not highly polarized (this is not clearly seen in the video) (**1:20:00**).

The experiment is repeated with a stack of hundreds of panes of glass. The reflected spot at the side is seen to be 100% polarized (as indeed it was with one pane of glass). The transmitted beam is shown to also be 100% polarized (again not clearly seen in the video). The stack of glass plates at the Brewster angle converts unpolarized light into a reflected beam to the side, which is 100% polarized in the perpendicular direction, and a transmitted beam which is 100% polarized in the parallel direction. Physics would seem to make the impossible possible, and difficult things easy!

8.03SC Physics III: Vibrations and Waves
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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.