Massachusetts Institute of Technology OpenCourseWare

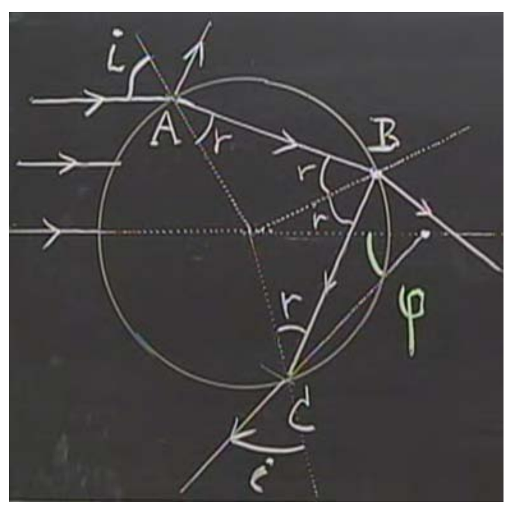
8.03SC **Fall 2012**

Notes for Lecture #22: Rainbows

Having completed an extensive discussion of the properties of electromagnetic waves, you are ready to tackle a fairly complex physical problem. Many people have *looked* at a rainbow, but like fine art it needs a trained eye to really *see* one. You now have that trained eye. Before proceeding to apply your training, consider this list of properties you may have noticed in looking at rainbows as well as questions that would be interesting to analyze and understand in detail:

1. Is red on the outside or inside (what is the order of colors)?
2. What is the angular radius (in degrees)?
3. What is the length?
4. What is the width (angular width in degrees)?
5. How does the light intensity inside the bow compare with that outside it?
6. At what time of day are rainbows seen?
7. In which direction, N, E, S, W, are they seen?
8. Are there 1, 2, or 3 bows?
9. If there are 2 bows, where is the second one?
10. If there are 2, what is the color sequence in the second?
11. What would be the angular radius of a second bow?
12. What would be the angular width of a second bow?
13. Are the bows polarized?
14. If polarized, what is the direction of polarization?
15. Would they be weakly or strongly polarized?

Based on memory alone, most people can answer relatively few of these questions (**3:40**).



It is well known that rainbows originate in the interaction of light with rain drops, which can to a good approximation be considered as spherical. Trace a ray coming into a spherical water drop in a plane containing the center of the drop. Rays will have various incidence angles but we consider one ray at incidence angle i (note that we called the comparable angle θ_1 before). Inside the drop, denote the angle of refraction r. From Snell's Law, $n_1 \sin i = n_2 \sin r$. For air, $n_1 \approx 1$ to a high degree of accuracy. For water, $n_2 \approx 1.336$, averaged over all frequencies of visible colors. Light is reflected at the point of entry A but consider only the refracted light that enters the drop.

Similarly, at point B on the far side, some light will exit but, at this point, consider only the light that is reflected and stays inside the drop. After this reflection, there is further reflection and refraction at point C, but now consider only the refracted light which leaves the drop. Note that because of the spherical geometry, the angle of the light with respect to the normal to the surface is also r at points B and C, and therefore, the light emerges at the same angle i relative to the normal with which it entered the drop. The angle between the refracted beam which emerges from the drop at C and the direction of the incident light is denoted φ (**5:35**).

From the geometry, it can be shown that $\varphi = 4r - 2i$. If light penetrates the drop normal to the surface, i.e. on a line pointing towards the center, it will come straight back out, and so $\varphi = 0$. What is not so intuitive is that φ has a maximum value. Clearly, when i is small, increasing it will increase both r and φ. However, φ eventually reaches a maximum for an incidence angle i of about 60°. The angle i increases as the distance of the incoming ray from the center of the drop increases: however, eventually a point is reached where further motion away from the center results in φ decreasing. Using Snell's Law to calculate r, the values of $\varphi = 4r - 2i$ in degrees are:

i	0	10	20	30	40	50	60	70	80
r	0	7.5	14.8	22.0	28.7	35.0	40.4	44.7	47.4
φ	0	10	19.2	28.0	34.8	40.0	41.6	38.8	29.6

As expected, when i is small and increasing, r and φ also increase. However, for i larger than about 60°, r keeps increasing but φ decreases. This is a key point in the formation of a rainbow. To find the exact incident angle at which φ becomes maximal, use Snell's law to relate r and i, set $\frac{d\varphi}{di} = 0$ and solve for $i_{\varphi_{max}}$. The result for an index of refraction n is $\cos^2(i_{\varphi_{max}}) = \frac{n^2-1}{3}$ (**9:00**). The value of φ_{max} itself can be found using $i_{\varphi_{max}}$, Snell's law, and $\varphi = 4r - 2i$.

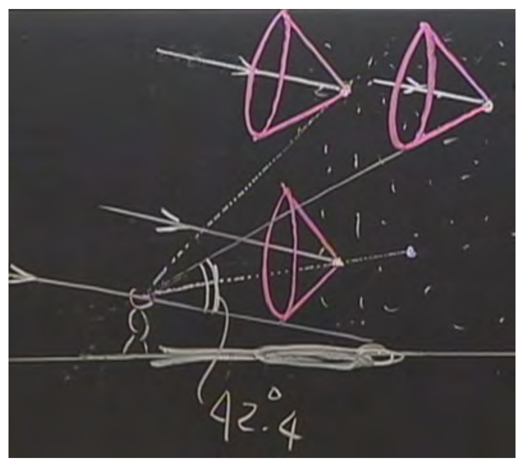
Since n varies slightly with frequency, these angles will also vary, and this effect is what forms the spread-out colors of a rainbow. Taking the extreme visible colors of red and violet, the results listed below are found. Violet light has a larger n and thus travels slower in water than red light.

	n	$i_{\varphi_{max}}$	$r_{\varphi_{max}}$	φ_{max}
Red light	1.331	59.53°	40.36°	42.37°
Violet light	1.343	58.83°	39.58°	40.65°

When sunlight containing all colors hits a droplet as parallel beams from the left (as shown in the figure), a cone of light will go back toward the left (**12:00**). Red light can come back at any angle up to 42.4° away from the incident direction. Violet light, on the other hand, can only come back at angles up to 40.7°. Imagine a screen with a small hole in it, with white light passing through the hole and hitting water drops. That light would come back to the screen as a circular pattern, with red light at the largest angle from the center and violet light at the smallest angle. All visible light can come back at angles less than 40.7°, so anywhere in the interior region the mix of colors will appear white. Finally, *no* light can come back at angles greater than for red light, so the outside region would appear dark.

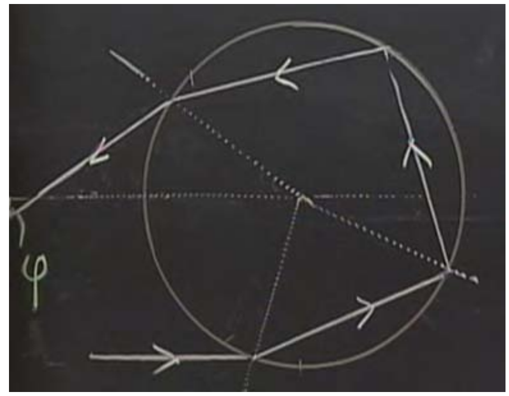
Rainbow colors appear very clear and pure. This may not be surprising for red: only red light can be bent to this large angle. However, why is the violet light also apparently very pure, since other colors could also be bent to that smaller angle? Why does it not appear washed out? The answer lies in the complicated calculation of intensities using the Fresnel equations (**16:00**). The intensity for any given color is sharply peaked at its maximal angle, while the remaining colors scattered to that angle are not, and therefore the other colors make a relatively small contribution. In the middle area, all of the colors being present causes a white color but very dim for this same reason.

There is no screen when we look at a rainbow. Why do we see it in the way that we do? If there are many drops in the opposite direction from a sunbeam coming past you (to have such a sunbeam, there must be clear sky behind you) then each will send back light toward you in the manner discussed above. If you are facing away from the Sun, the shadow of your head will be in exactly the opposite direction to the Sun. A droplet located at a larger angle than 42° from the direction away from the Sun (antisolar) will send all of its light back inside a cone which you are not in. Thus the region of sky at > 42° will be dark (**19:30**). Conversely, droplets located at a small angle from the antisolar direction will send back white light. Those droplets at exactly 42° from the antisolar direction will send red light toward you. The droplets satisfying this condition are located on a cone centered on you. The whole thing is axially symmetric, so looking at the horizon you will see circular bows centered on the shadow of your head. The same will apply to other colors, whose cones appear at the appropriate angles, thus forming a rainbow from the individual effects of a huge number of droplets (**22:55**).



If there are drops close to you, the bow can extend below the horizon. In all cases, the center of the rainbow is the same angle below the horizon as the Sun is above it. If the Sun is on the horizon, the rainbow will be as high in the sky, and hence as large as possible. If the Sun is high in the sky, the center of the rainbow will be farther down and there will be less of the rainbow above the horizon. This explains why rainbows are preferentially seen near sunrise and sunset, and cannot be seen (at least not to have any part above the horizon) if the Sun is higher than 42° in the sky.

Nature produces two bows, called primary and secondary. The secondary is formed through two internal reflections, rather than the single one discussed above. These drawings are symmetric and so there is no difference in the physics for light entering the top or the bottom. However, notice that for light reflecting *downward* relative to its initial direction, as is typically the case for rainbows seen in the sky, light for the primary must enter near the top, while that for the secondary enters near the bottom.



Following similar geometrical arguments, the secondary emerges where φ is a *minimum* (φ_{min}) (**26:15**), specifically 50.37° for red light, and 53.47° for violet. therefore, the secondary rainbow is outside the primary, i.e. at a larger angle, and the colors are reversed with red *inside* of violet. In addition, the secondary is considerably fainter since additional light is lost at the extra reflection.

We can now "see" rainbows with new eyes. First, note that all of the above is based on geometric optics: only Snell's Law, ray paths, and reflection/refraction using Fresnel's equations where needed. Already Isaac Newton in his book Opticks (1704) could explain the primary and secondary rainbows by considering the path of rays through drops much as we have done.

You can make your own rainbow even with the sun high overhead by standing in a garden sprinkler so that there are water drops below you. This gives a rainbow covering a full 360° circle, surrounded by the shadow of your head. Rainbows found in art are sometimes not depicted very accurately. An example shows an arc of only two colors, with red on the inside, the opposite of the actual ordering. A photograph of an artificial rainbow made using water sprayed from a garden hose clearly shows the predicted color order, as well as white light in the middle, and the darker area outside (**31:15**). Sometimes one has to suffer to do interesting science. The photos were taken in January, when working outside with water can make one rather cold. This example does not clearly show the secondary rainbow, largely because the background is too bright for this fainter arc to be seen. A photograph against the dark backdrop of an asphalt driveway does show the secondary, further out, fainter, and with reversed colors as expected. A very dramatic photo of a rainbow above the VLA radio telescope in Socorro, New Mexico, shows not only the white interior region and the dark region between the two bows, but also a slightly brighter region outside the secondary. Recall that the secondary is due to a *minimum* in φ, so that light can come back from larger angles (**35:00**).

The geometric optics governing rainbow formation applies if the water drops are very large compared to the wavelength of light, say a few mm. For drops smaller than this, say about a half mm or smaller, diffraction can lead to either constructive or destructive interference. One of the effects can be dark bands, due to destructive interference, inside the primary, leading to so-called *supernumary* bows, another effect visible in the VLA photo.

Once one has quite small drops, say 100 μm (0.1 mm) or smaller, diffraction begins to plays a dominant role, eventually totally mixing the colors for drops smaller than about 50 μm. Such rainbows are rare but an example taken in a mist in the Arctic is shown. This effect cannot be due to ice crystals since the spherical symmetry of water droplets is required to get the curved shape of rainbows. The dominance of diffraction also results in a strong supernumary effect. Another rare type of rainbow is formed in the red light near sunrise or sunset. A rainbow formed under such circumstances has a purely red primary arc and also a purely red area inside the primary (**39:30**).

Except for polarization, we now know the answers to the questions posed at the start of the lecture but there are a few subtleties. For example, the angular width of a rainbow is about 0.5° larger than would be inferred from the angles given above due to the size of the Sun. The degree of polarization is observed to be very large, 91% (**43:30**). Looking at the geometry of rainbow formation, this should not be surprising. As discussed in previous lectures, reflected light is strongly polarized in a direction perpendicular to the plane containing the incident and reflected beams. This polarization approaches 100% near the Brewster angle. For reflections at a water-to-air interface, and using an average value for n over all colors, the Brewster angle ($\tan_{Br} = n_2/n_1$) is 36.8°. From the tables given above, this is very close to the reflection angle r at which the rainbow light is at its highest intensity. From the geometry, a direction perpendicular to the plane of incidence is parallel (i.e. tangential) to the curve of the rainbow. Coincidentally, polarized sunglasses, set up to block the horizontal polarization typical of glare reflections at a road or water surface, will also block much of a rainbow (**47:40**).

A rainbow demonstration is done by shining light on a glass globe filled with water (**50:30**). Being pedantic, this is not a *rain*bow but, rather, is identical to light from a single drop. All of the properties of real rainbows, including the polarization, can be observed. Even the white light in the inner region is due to reflection close to the Brewster angle, so it is also highly polarized.

There are other optical phenomena in the sky that are quite common. Ice crystals can cause haloes around bright objects, the most common being a 22° halo around the Sun or Moon. Although the colors are generally subdued in these haloes, red is on the inside (**54:30**). There are many other phenomena, with various names, associated with ice crystals in the atmosphere including a second 46° halo and so-called sundogs.

The phenomenon called a glory is due to diffraction (as opposed to reflection and refraction) from extremely fine water drops. Like a rainbow, it is seen in the antisolar direction, most commonly from looking down into the clouds from an airplane. The glory appears as multicolored circles around the shadow of the airplane. As a diffraction phenomenon, its angular size is inversely related to the size of the diffracting object: smaller droplets give a larger glory (**57:30**). When looking away from the Sun into a wall of fog, one can see a glory which looks like a halo around the shadow of your head (**1:00:35**).

Now, consider the image shown at the beginning of the lecture videos. This was used as an "Astronomy Picture of the Day", and out of about 3000 responses, about 50 had some idea of what was going on but only 5 showed a complete understanding of the physics. Some guesses as to what it was were rather farfetched. From the color ordering and white interior, one could infer that small transparent spheres are involved but the angular radius is roughly 20° and the width of the colored band is about 16% of the radius, whereas with water drops, one would expect about 40° and 5%, respectively. This photo was taken at a construction site in June, with the Sun high in the sky and the "rainbow" is due to spherical transparent beads used in sandblasting which had spilled on the ground (**1:06:00**).

The identical analysis discussed above can be used to find the angular radius of such a "glassbow". Using an appropriate index of refraction (n=1.5), the angular radius is found to be 22.8°. From the geometry of the photo and assuming a 20 cm head diameter and head height of about 5 feet (1.5 m), one can estimate the observed angular radius to be about 20°. As for rainbows, the Brewster angle is very close to the internal reflection angle and the "sandbow" is also highly polarized (**1:10:15**). Using similar beads glued to black paper, illuminated by a bright light source, all of the expected features, including the degree and direction of polarization can be demonstrated.

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These viewing notes were written by Prof. Martin Connors in collaboration with Dr. George S.F. Stephans.