Problem Set #1
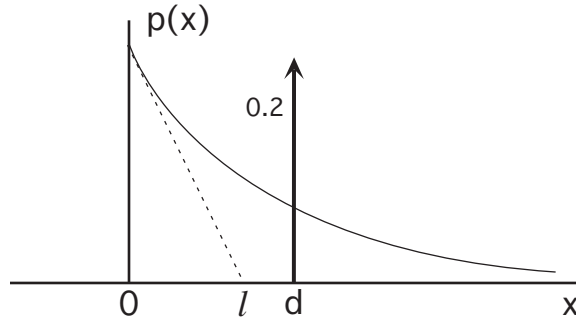
Due in hand-in box by 12:40 PM, Wednesday, February 13

Problem 1: Doing a Semiconductor

After diffusing impurities into a particular semiconductor the probability density $p(x)$ for finding a given impurity a distance x below the surface is given by

$$
\begin{aligned}
p(x) &= (0.8/l)\exp[-x/l] + 0.2\,\delta(x-d) & x \geq 0 \\
&= 0 & x < 0
\end{aligned}
$$

where l and d are parameters with the units of distance. The delta function arises because a fraction of the impurities become trapped on an accidental grain boundary a distance d below the surface.

a) Make a carefully labeled sketch of the cumulative function $P(x)$ which displays all of its important features. [You do <u>not</u> need to give an analytic expression for $P(x)$.]

b) Find $< x >$.

c) Find the variance of x, $\mathrm{Var}(x) \equiv < (x- < x >)^2 >$.

The contribution to the microwave surface impedance due to an impurity decreases exponentially with its distance below the surface as $e^{(-x/s)}$. The parameter s, the "skin depth", has the units of distance.

d) Find $< e^{(-x/s)} >$.

Problem 2: A Peculiar Probability Density
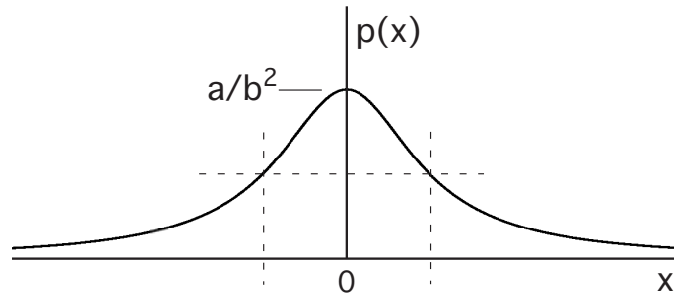
Consider the following probability density.

$$p(x) = \frac{a}{b^2 + x^2}$$

The functional form is variously called a Lorentzian or a Cauchy density. In physics, many spectral lines associated with resonance phenomena can be approximated by this function where x is replaced by a radian, ω, or circular, ν, frequency.

a) Use normalization to find a as a function of b.

b) Find the cumulative function $P(x)$ and sketch the result.

c) Find $< x >$.

d) Find the values at which the density falls to one half of its maximum value, its half-width at half-height.

e) What happens to $< x^2 >$ and $\text{Var}(x)$ for this density?

Problem 3: Visualizing the Probability Density for a Classical Harmonic Oscillator

Take a pencil about 1/3 of the way along its length and insert it between your index and middle fingers, between the first and second knuckles from the end. By moving those fingers up and down in opposition you should be able to set the pencil into rapid oscillation between two extreme angles. Hold your hand at arms length and observe the visual effect. We will examine this effect.

Consider a particle undergoing simple harmonic motion, $x = x_0 \sin(\omega t + \phi)$, where the phase ϕ is completely unknown. The amount of time this particle spends between x and $x + dx$ is inversely proportional to the magnitude of its velocity (its speed) at x. If one thinks in terms of an ensemble of similarly prepared oscillators, one comes to the conclusion that the probability density for finding an oscillator at x, $p(x)$, is proportional to the time a given oscillator spends near x.

    a) Find the speed at x as a function of x, ω, and the fixed maximum displacement x_0.

    b) Find $p(x)$. [Hint: Use normalization to find the constant of proportionality.]

    c) Sketch $p(x)$. What are the most probable values of x? What is the least probable? What is the mean (no computation!)? Are these results consistent with the visual effect you saw with the oscillating pencil?

Problem 4: Quantized Angular Momentum

In a certain quantum mechanical system the x component of the angular momentum, L_x, is quantized and can take on only the three values $-\hbar$, 0, or $\hbar$. For a given state of the system it is known that $< L_x > = \frac{1}{3}\hbar$ and $< L_x^2 > = \frac{2}{3}\hbar^2$. [$\hbar$ is a constant with units of angular momentum. No knowledge of quantum mechanics is necessary to do this problem.]

    a) Find the probability density for the x component of the angular momentum, $p(L_x)$. Sketch the result.

    b) Draw a carefully labeled sketch of the cumulative function, $P(L_x)$.

Problem 5: A Coherent State of a Quantum Harmonic Oscillator

In quantum mechanics, the probability density for finding a particle at a position $\vec{r}$ at time t is given by the squared magnitude of the time dependent wavefunction $\Psi(\vec{r},t)$:

$$p(\vec{r},t) = |\Psi(\vec{r},t)|^2 = \Psi^*(\vec{r},t)\Psi(\vec{r},t).$$

Consider a particle moving in one dimension and having the wavefunction given below [yes, it corresponds to an actual system; no, it is not indicative of the simple wavefunctions you will encounter in 8.04].

$$\Psi(x,t) = (2\pi x_0^2)^{-1/4} \exp\left[-\frac{i\omega t}{2} - \frac{i}{(2x_0)^2}(2\alpha x x_0 \sin\omega t - \alpha^2 x_0^2 \sin 2\omega t) - (\frac{x - 2\alpha x_0 \cos\omega t}{2x_0})^2\right]$$

x_0 is a characteristic distance and α is a dimensionless constant.

a) Find the expression for $p(x,t)$.

b) Find expressions for the mean and the variance [Think; don't calculate].

c) Explain in a few words the behavior of $p(x,t)$. Sketch $p(x,t)$ at $t = 0$, $\frac{1}{4}T$, $\frac{1}{2}T$, $\frac{3}{4}T$, and T where $T \equiv 2\pi/\omega$.

Problem 6: Bose-Einstein Statistics

You learned in 8.03 that the electro-magnetic field in a cavity can be decomposed (a 3 dimensional Fourier series) into a countably infinite number of modes, each with its own wavevector $\vec{k}$ and polarization direction $\vec{\epsilon}$. You will learn in quantum mechanics that the energy in each mode is quantized in units of $\hbar\omega$ where $\omega = c|\vec{k}|$. Each unit of energy is called a photon and one says that there are n photons in a given mode. Later in the course we will be able to derive the result that, in thermal equilibrium, the probability that a given mode will have n photons is

$$p(n) = (1 - a)a^n \qquad n = 0,\ 1,\ 2,\ \cdots$$

where $a < 1$ is a dimensionless constant which depends on ω and the temperature T. This is called a Bose-Einstein density by physicists; mathematicians, who recognize that it is applicable to other situations as well, refer to it as a geometric density.

a) Find $<n>$. [Hint: Take the derivative of the normalization sum, $\sum a^n$, with respect to a.]

b) Find the variance and express your result in terms of $<n>$. [Hint: Now take the derivative of the sum involved in computing the mean, $\sum na^n$.] For a given mean, the Bose-Einstein density has a variance which is larger than that of the Poisson by a factor. What is that factor?

c) Express $p(x)$ as an envelope function times a train of δ functions of unit area located at the non-negative integers. Show that the envelope decreases exponentially, that is, as $e^{-x/\phi}$. Express ϕ in terms of $< n >$ and show that in the limit of large $< n >$, $\phi \rightarrow < n >$.

8.044 Statistical Physics I
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