MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2013

Solutions to Problem Set #8

Problem 1: Dust Grains in Space

a) $\mathcal{H}$ is separable: the 6 variables are statistically independent.

$$p(\theta_1, \theta_2, \theta_3, L_1, L_2, L_3) =$$

$$\underline{\left(\frac{1}{2\pi}\right)\left(\frac{1}{2\pi}\right)\left(\frac{1}{2\pi}\right)\frac{1}{\sqrt{2\pi I_1 kT}}\exp(-L_1^2/2I_1kT)\frac{1}{\sqrt{2\pi I_2 kT}}\exp(-L_2^2/2I_2kT)\frac{1}{\sqrt{2\pi I_3 kT}}\exp(-L_3^2/2I_3kT)}$$

b)
$$< L_1^2 >=< L_2^2 >= I_1kT \quad \gg \quad < L_3^2 >= I_3kT$$

$\Rightarrow \vec{L}$ is almost $\perp$ to axis 3, the long axis.

c)
$$Z_R = (Z_{1,R})^N = \left[(2\pi)^{9/2}\sqrt{I_1I_2I_3}(kT)^{3/2}\right]^N$$

$$\begin{aligned}
F_R &= -NkT\ln Z_{1,R} \\
S_R &= -\left(\frac{\partial F_R}{\partial T}\right)_N = Nk\ln Z_{1,R} + NkT\,\frac{1}{Z_{1,R}}\,\frac{3}{2}\,\frac{1}{T}\,Z_{1,R} \\
&= \underline{Nk\ln Z_{1,R} + \frac{3}{2}Nk}
\end{aligned}$$

d) 3^{rd} law is violated: $\lim_{T\to 0} S_R = Nk\ln(0) = -\infty$. At very low temperatures one must switch to a quantum treatment of the rotational motion. Such a treatment will lead to a result consistent with the 3^{rd} law.

e) There is no energy gap behavior because there is no gap in the classically allowed rotational energies. The quantum result, however, will show an energy gap.

Problem 2: Adsorption On a Stepped Surface

a) $Z_1 = \underline{\sum_{\text{states}} \exp(-\epsilon_{\text{state}}/kT) = 0.01M + 0.14M \exp(-\Delta/kT) + 0.85M \exp(-1.5\Delta/kT)}$

b)

$$\frac{n_{\text{face}}}{n_{\text{corner}}} = \frac{p_{\text{face}}}{p_{\text{corner}}} = \frac{0.85M \exp(-1.5\Delta/kT)}{0.01M} = \underline{85 \exp(-1.5\Delta/kT)}$$

c) Consider only the 2 lowest energy levels

$$\begin{aligned} E &= N < \epsilon_{\text{one}} > \\ &= N \left[(0) \frac{0.01M}{0.01M + 0.14M \exp(-\Delta/kT)} + (\Delta) \frac{0.14M \exp(-\Delta/kT)}{0.01M + 0.14M \exp(-\Delta/kT)} \right] \\ &\approx 14N\Delta \exp(-\Delta/kT) \end{aligned}$$

$$C_A = \left(\frac{\partial E}{\partial T}\right)_A = 14N\Delta \left(\frac{\Delta}{kT^2}\right) \exp(-\Delta/kT) = \underline{14Nk \left(\frac{\Delta}{kT}\right)^2 \exp(-\Delta/kT)}$$

d) All states are equally likely $\Rightarrow \underline{p_{\text{face}} = 0.85}$.

e) M possible states for each atom $\Rightarrow \underline{\lim_{T\to\infty} S = Nk \ln M}$.

f) One expects energy gap behavior because there is an energy gap for the excitation of a single atom.

Problem 3: Neutral Atom Trap

a) First write down the Hamiltonian for one atom.

$$\mathcal{H}_1 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + ar$$

Then compute the partition function

$$\begin{aligned} Z_1 &= \frac{1}{h^3} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{p_x^2}{2mkT}}\, dp_x}_{\sqrt{2\pi mkT}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{p_y^2}{2mkT}}\, dp_y}_{\sqrt{2\pi mkT}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{p_z^2}{2mkT}}\, dp_z}_{\sqrt{2\pi mkT}} \underbrace{\int_V e^{-\frac{ar}{kT}}\, r^2 \sin\theta\, dr d\theta d\phi}_{4\pi \int_0^\infty \exp[-ar/kT]\, r^2\, dr} \\ &= (\frac{2\pi mkT}{h^2})^{3/2} 4\pi \left(\frac{kT}{a}\right)^3 \underbrace{\int_0^\infty y^2 e^{-y}\, dy}_{2} \\ &= 8\pi k^3 (\frac{2\pi mk}{h^2})^{3/2}\, T^{9/2} a^{-3} \end{aligned}$$

In order to emphasize the dependence on the important variables, this can be written in the form $Z_1 = AT^\alpha a^{-\eta}$ where

$$\underline{A = 8\pi k^3 (\frac{2\pi mk}{h^2})^{3/2}} \quad \underline{\alpha = 9/2} \quad \text{and } \underline{\eta = 3}.$$

b) Remember to include correct Boltzmann counting.

$$\begin{aligned} Z &= \frac{1}{N!} Z_1^N \\ F &= -kT \ln Z = -kT(N \ln Z_1 - N \ln N + N) \\ &= -NkT \ln(Z_1/N) - NkT \\ S &= -\left(\frac{\partial F}{\partial T}\right)_N \\ &= Nk \ln(Z_1/N) + Nk + NkT \frac{1}{Z_1/N} (9/2) \frac{Z_1/N}{T} \\ &= \underline{Nk \ln(Z_1/N) + (11/2)Nk} \end{aligned}$$

c)

$$dQ = 0 \qquad \text{no heat is exchanged with surroundings}$$

$$dQ = dS/T \qquad \text{process is said to be reversible}$$

$$\Rightarrow \; dS = 0, \quad S \text{ is constant}$$

$$\Rightarrow \; Z_1 \text{ is constant, using the result from b)}$$

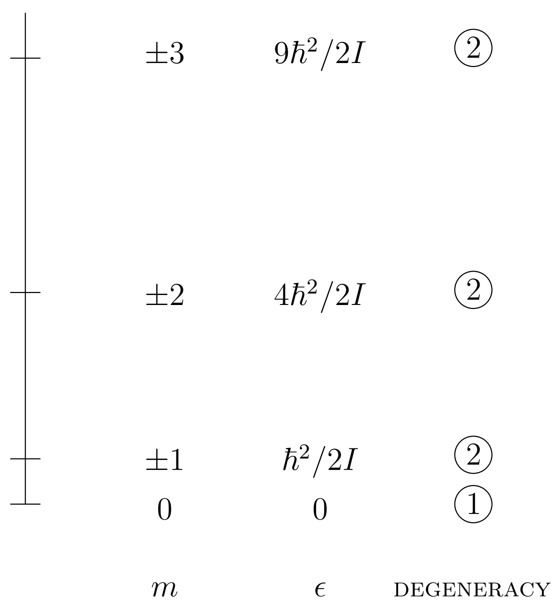
$$\Rightarrow \; T^{9/2}/a^3 \text{ is constant and } = T_0^{9/2}/a_0^3$$

$$\left(\frac{T}{T_0}\right)^{9/2} = \left(\frac{a}{a_0}\right)^3$$

$$T = \underline{T_0 \left(\frac{a}{a_0}\right)^{2/3}}$$

Problem 4 Two-Dimensional H_2 Gas

a)



$$\epsilon_m = (\hbar^2/2I)\, m^2$$

$$Z_{\text{ROT},1} = \sum_{states} \exp[-\epsilon(\text{state})/kT] = \sum_{m=-\infty}^{\infty} \exp[-(\hbar^2/2IkT)\, m^2\,]$$

$$= \underline{1 + 2\sum_{j=1}^{\infty} \exp[-(\hbar^2/2IkT)\, j^2\,]}$$

b)

$$\frac{p(m=3)}{p(m=2)} = \frac{Z^{-1}\exp[-9\hbar^2/2IkT]}{Z^{-1}\exp[-4\hbar^2/2IkT]} = \exp[-(5/2)\hbar^2/IkT]$$

c)

$$p(m=3\,|\,\epsilon = 9\hbar^2/2I) = \frac{Z^{-1}\exp[-9\hbar^2/2IkT]}{2(Z^{-1}\exp[-9\hbar^2/2IkT])} = \underline{1/2}$$

$$p(m=1\,|\,\epsilon \leq \hbar^2/2I) = \frac{Z^{-1}\exp[-\hbar^2/2IkT]}{Z^{-1}+2(Z^{-1}\exp[-\hbar^2/2IkT])} = \underline{\frac{1}{2+\exp[\hbar^2/2IkT])}}$$

d)

$$\begin{aligned}
Z_{\mathrm{ROT},1} &= \sum_{m=-\infty}^{\infty}\exp[-(\hbar^2/2IkT)\,m^2\,] \to \int_{-\infty}^{\infty}\exp[-(\hbar^2/2IkT)\,m^2\,]\,dm \\
&= \int_{-\infty}^{\infty}\exp[-\frac{m^2}{2(IkT/\hbar^2)}]\,dm = \underbrace{\left(\frac{2\pi IkT}{\hbar^2}\right)^{1/2}}_{\text{Gaussian normalization}} \propto \beta^{-1/2} \\
E_{\mathrm{ROT}} &= N<\epsilon>= N\left(-\frac{1}{Z_1}\frac{\partial Z_1}{\partial\beta}\right) \\
&= N\left(-\frac{1}{Z_1}\right)\left(-\frac{Z_1}{2\beta}\right) = (1/2)N\frac{1}{\beta} = \underline{(1/2)NkT}
\end{aligned}$$

Problem 5: Why Stars Shine

a) The electostatic potential outside the charged sphere depends only on r, the magnitude of the distance from the center of the sphere.

$$\phi(r) = \frac{|e|}{r} \qquad r \geq R$$

The potential energy of another proton, considered to be a point particle, in this field is

$$V(r) = q\phi(r) = \frac{e^2}{r}$$

Then the minimum energy that the second proton must have to get within a radial distance R of the first is

$$E_{min} = V(R) = \frac{e^2}{R} = \frac{(4.8 \times 10^{-10})^2}{1.2 \times 10^{-13}} = \underline{1.92 \times 10^{-6} \text{ ergs}}$$

b) In problem 4 of problem set 2 we found the following expression for the kinetic energy of a particle in a three dimensional classical gas.

$$p(E) = \frac{2}{\sqrt{\pi}} \frac{1}{kT} \sqrt{\frac{E}{kT}} \exp[-E/kT]$$

Now find the probability p_+ that a given proton in the stellar plasma has an energy greater than E_{min}.

$$\begin{aligned}
p_+ &\equiv \text{prob}(E > E_{\text{min}}) = \int_{E_{\text{min}}}^{\infty} p(E)\, dE \\
&= \frac{2}{\sqrt{\pi}} \frac{1}{kT} \int_{E_{\text{min}}}^{\infty} \sqrt{\frac{E}{kT}} \exp[-E/kT]\, dE = \frac{2}{\sqrt{\pi}} \int_{y_{\text{min}}=E_{\text{min}}/kT}^{\infty} \sqrt{y} \exp[-y]\, dy \\
&\approx \frac{2}{\sqrt{\pi}} \sqrt{y_{\text{min}}} \exp[-y_{\text{min}}]
\end{aligned}$$

This is going to turn out to be a very small number, probably too small to be represented on a hand calculator. Therefore, let's work toward getting its logarithm.

$$\begin{aligned}
\log_{10}(p_+) &= \log_{10}\left[\frac{2}{\sqrt{\pi}} \sqrt{\frac{E_{\text{min}}}{kT}}\right] + \log_{10}\left[\exp[-E_{\text{min}}/kT]\right] \\
\log_{10}\left[\exp[-E_{\text{min}}/kT]\right] &= -\frac{E_{\text{min}}}{kT} \log_{10}(e) = -(.4343)\frac{E_{\text{min}}}{kT} \\
\frac{E_{\text{min}}}{kT} &= \frac{1.920 \times 10^{-6}}{1.381 \times 10^{-16} \times 4 \times 10^7} = 3.476 \times 10^2 \\
\log_{10}(p_+) &= 1.323 - 1.510 \times 10^2 = -149.6 \\
p_+ &= \underline{0.2 \times 10^{-149}}
\end{aligned}$$

c)

$$
\begin{aligned}
< v > &= \sqrt{\frac{8kT}{\pi m}} \\
&= \left(\frac{8 \times 1.381 \times 10^{-16} \times 4 \times 10^{7}}{\pi \times 1.67 \times 10^{-24}} \right)^{1/2} = \underline{9.18 \times 10^{7} \text{ cm/sec}} \\
\sigma &= \pi (2R)^2 = \pi (2.4 \times 10^{-13})^2 = 1.81 \times 10^{-25} \text{ cm}^2 \\
n &= \frac{\rho}{M_{\text{proton}}} = \frac{100}{1.67 \times 10^{-24}} = 5.99 \times 10^{25} \text{ protons/cm}^3 \\
L &= (n\sigma)^{-1} = \underline{9.22 \times 10^{-2} \text{ cm}} \\
\tau_{\text{collision}} &= L/ < v >= \underline{1.01 \times 10^{-9} \text{sec}}
\end{aligned}
$$

d) The fusion rate per proton is p_+ times the collision rate per proton. But in general a rate equals the reciprocal of the characteristic time between events, so

$$\tau_{\text{fusion}} = \tau_{\text{collision}}/p_+ = \frac{1.01 \times 10^{-9}}{0.2 \times 10^{-149}} = \underline{5 \times 10^{140} \text{sec}}$$

The universe is about 15 billion years old, corresponding to a time

$$T_{\text{universe}} = 15 \times 10^9 \times 365 \times 24 \times 60 \times 60 = 4.7 \times 10^{17} \text{ sec}$$

If the mass of the sun is 2×10^{33} grams then the number of protons it contains is given by

$$N_{\text{protons}} = \frac{2 \times 10^{33}}{1.67 \times 10^{-24}} = 1.2 \times 10^{57}$$

Then for the entire sun, the total number of fusions per second is found as follows.

$$
\begin{aligned}
\text{number of fusions per second} &= N_{\text{protons}} \times \text{ fusion rate per proton} \\
&= N_{\text{protons}}/\tau_{\text{fusion}} \\
&= 1.2 \times 10^{57} \;/\; 5 \times 10^{140} = \underline{2 \times 10^{-84} \text{ sec}^{-1}}
\end{aligned}
$$

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