MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2013

Solutions to Problem Set #9

Problem 1: The Big Bang

If the expansion is adiabatic, $\Delta S = 0$.

$$
\begin{aligned}
S &= -\left(\frac{\partial F}{\partial T}\right)_V \\
&= -\frac{\partial}{\partial T}\left(-\frac{1}{45}\frac{\pi^2}{c^3\hbar^3}(kT)^4 V\right) \\
&= \frac{4}{45}\frac{\pi^2}{c^3\hbar^3}k^4T^3V
\end{aligned}
$$

From this result we see that the product T^3V remains constant during the expansion. Therefore

$$\frac{V(3\mathrm{K})}{V(3000\mathrm{K})} = \frac{3000^3}{3^3} = \underline{10^9}$$

Problem 2: Lattice Heat Capacity of Solids

a) The heat capacity of a classical harmonic oscillator is k, independent of its frequency.

# of oscillators = (3 degrees of freedom) × (J atoms/unit cell) × (N unit cells) = $3JN$.

Therefore, $\underline{C_V = 3JNk}$.

b) For a quantum harmonic oscillator

$$
\begin{aligned}
<\epsilon> &= h\nu(<n> + \frac{1}{2}) \\
<n> &= (\exp[h\nu/kT] - 1)^{-1}
\end{aligned}
$$

Assuming all oscillators have the same frequency

$$
\begin{aligned}
C_V &= 3JN\,\frac{\partial <\epsilon>}{\partial T} \\
&= 3JNh\nu\,\frac{\partial <n>}{\partial T} \\
&= 3JNh\nu\,(\frac{h\nu}{kT^2})\,\frac{\exp[h\nu/kT]}{(\exp[h\nu/kT]-1)^2}
\end{aligned}
$$

$$= \underline{3JNk\,(\frac{h\nu}{kT})^2 \frac{\exp[h\nu/kT]}{(\exp[h\nu/kT]-1)^2}}$$

$$\lim_{kT \ll h\nu} C_V = 3JNk\,(\frac{h\nu}{kT})^2 \exp[-h\nu/kT] \qquad \text{energy gap behavior}$$

$$\lim_{kT \gg h\nu} C_V = 3JNk \qquad \text{classical result}$$

c i)

$$E(T) = \int_0^\infty D(\omega) < \epsilon(\omega, T) > \, d\omega$$

$$= \underline{\int_0^\infty D(\omega) \left\{ \frac{1}{\exp[\hbar\omega/kT]-1} + \frac{1}{2} \right\} d\omega}$$

For $\hbar\omega \ll kT$, $< \epsilon > \to kT$ independent of ω. Therefore one can move it out from under the integral.

$$\lim_{kT \gg \hbar\omega_{\max}} E(T) = kT \underbrace{\int_0^\infty D(\omega)\, d\omega}_{3JN} = 3JNkT$$

$$\lim_{kT \gg \hbar\omega_{\max}} C_V(T) = \underline{3JNk} \qquad \text{the classical result}$$

c ii)

$$E(T) = \underbrace{\int_0^\infty D(\omega)(\hbar\omega/2)\, d\omega}_{E_0,\ \text{independent of } T} + \int_0^\infty \frac{\hbar\omega D(\omega)}{\exp[\hbar\omega/kT]-1}\, d\omega$$

For very low T one may use the low ω limiting form of $D(\omega)$ since only the oscillators with low ω will be excited.

$$E(T) = E_0 + \frac{3V}{2\pi^2} \frac{1}{< v >^3} \int_0^\infty \frac{\hbar\omega \cdot \omega^2}{\exp[\hbar\omega/kT]-1}\, d\omega$$

$$= E_0 + \frac{3V}{2\pi^2} \left(\frac{kT}{\hbar < v >} \right)^3 kT \int_0^\infty \frac{(\hbar\omega/kT)^3}{\exp[\hbar\omega/kT]-1}\, d(\hbar\omega/kT)$$

$$= E_0 + \frac{3V}{2\pi^2} \left(\frac{kT}{\hbar < v >} \right)^3 kT \underbrace{\int_0^\infty \frac{x^3}{e^x - 1}\, dx}_{\frac{\pi^4}{15}}$$

$$= \underline{E_0 + \frac{\pi^2}{10} V \left(\frac{kT}{\hbar < v >} \right)^3 kT}$$

$$C_V(T) = \underline{\frac{2}{5}\pi^2 V \left(\frac{kT}{\hbar < v >} \right)^3 k} \quad \propto \quad T^3$$

Note that the exponential temperature dependence associated with energy gap behavior has been washed out by the distribution of energy gaps associated with the distribution of harmonic oscillator frequencies, some less than kT at any reasonable value of the temperature.

Problem 3: Thermal Noise in Circuits I, Mean-Square Voltages and Currents

a) The energy stored in the capacitor, $E = \frac{1}{2}Cv^2$, acts as the Hamiltonian for this small subsystem. We can then apply the results of the canonical ensemble.

$$
\begin{aligned}
p(v) &\propto \exp[-\frac{1}{2}Cv^2/kT] \\
&= \left(2\pi\frac{kT}{C}\right)^{-1/2} \exp[-v^2/2(kT/C)] \qquad \text{when normalized}
\end{aligned}
$$

Now we do the numbers.

$$
\begin{aligned}
kT &= 1.4 \times 10^{-23} \times 300 = 4.2 \times 10^{-21} \text{ joules} \\
C &= 100 \text{ pF} = 10^{-10} \text{ F} \\
< v^2 > &= kT/C = 42 \times 10^{-12} \\
\sqrt{< v^2 >} &= \sqrt{kT/C} = 6.5 \times 10^{-6} = \underline{6.5\ \mu\text{V}}
\end{aligned}
$$

b) Now the energy stored in the inductor, the effective Hamiltonian for the subsystem, is $E = \frac{1}{2}Li^2$.

$$
\begin{aligned}
p(i) &\propto \exp[-\frac{1}{2}Li^2/kT] \\
&= \left(2\pi\frac{kT}{L}\right)^{-1/2} \exp[-i^2/2(kT/L)] \qquad \text{when normalized}
\end{aligned}
$$

Again we do the numbers.

$$
\begin{aligned}
L &= 1 \text{ mH} = 10^{-3}\text{H} \\
< i^2 > &= kT/L = 4.2 \times 10^{-18} \\
\sqrt{< i^2 >} &= \sqrt{kT/L} = 2 \times 10^{-9}\text{A} = \underline{2\text{nA}}
\end{aligned}
$$

c) This method does not work for a resistor since it does not store energy; rather, it is completely dissipative. One can not, for example, write a Hamiltonian which describes only the voltage and current associated with the resistor. Of course, if the R were in parallel with a C or in series with an L one could make use of the above results. These are two particularly simple examples of the general result that the thermal voltage one would measure across a resistor (RMS) depends on the circuit to which it is connected. Note that this is not the case for a C or an L as shown above.

Problem 4: Thermal Noise in Circuits II, Johnson Noise of a Resistor

a) For a line of length L which is short circuited at each end,

$$\begin{aligned} E_n(x,t) &= E_n \sin \underbrace{(n\pi x/L)}_{k_n x} \sin(\omega_n t + \phi_n) \\ k_n &= \underline{n(\pi/L) \qquad n = 1, 2, 3, \cdots} \end{aligned}$$

Then $\omega_n = ck_n$ and ϕ_n is some fixed time phase factor. Only one polarization direction is allowed on a transmission line. For a coaxial cable $\vec{E}$ is always in the radial direction.

b)

$$\begin{aligned} \#(\omega < \omega_0) &= \#(k < \omega_0/c) = (\omega_0/c)/(\pi/L) = \frac{\omega_0 L}{\pi c} \\ D(\omega) &= \frac{d\#(\omega)}{d\omega} = \frac{L}{\pi c} \quad \text{a constant} \end{aligned}$$

c) Each mode is an independent harmonic oscillator, so

$$\begin{aligned} < \epsilon_n > &= \hbar\omega \left(\exp[\hbar\omega/kT] - 1\right)^{-1} + \frac{1}{2}\hbar\omega \\ u(\omega, T) &= \frac{D(\omega) < \epsilon_n(\omega) >}{L} \\ &= \left(\frac{L}{\pi c}\right) \frac{1}{L} \left(\hbar\omega \left(\exp[\hbar\omega/kT] - 1\right)^{-1} + \frac{1}{2}\hbar\omega\right) \end{aligned}$$

d) When $kT \gg \hbar\omega$, $< \epsilon_n > \rightarrow kT$. Then

$$u(\omega, T) \rightarrow \left(\frac{L}{\pi c}\right) \frac{1}{L}(kT) = \underline{\frac{kT}{\pi c}}$$

Note that this result for the double energy density (per unit length, per unit frequency interval) on the transmission line is independent of the frequency in this frequency region (low frequencies).

e) Consider the standing wave modes to be made up of two waves propagating in opposite directions. Then the thermal energy flow in each direction is

$$\frac{c}{2}u(\omega, T) = \frac{kT}{2\pi}.$$

If this flows from the line into the resistor, then by detailed balance in thermal equilibrium an equal amount must flow out:

$$\underline{P_n(\omega) = \frac{kT}{2\pi}}.$$

f)

$$\begin{aligned}
\Delta\nu &= 10 \text{ MHz} = 10^7 \text{ s}^{-1} \\
\Delta\omega &= 2\pi\Delta\nu = 2\pi \times 10^7 \text{ s}^{-1} \\
P_n(\omega)\Delta\omega &= \frac{kT}{2\pi} \times 2\pi \times 10^7 \text{ s}^{-1} \\
&= 1.38 \times 10^{-16} \text{ergs/K} \times 300 \text{ K} \times 10^7 \text{ s}^{-1} \\
&= \underline{4.1 \times 10^{-7} \text{ ergs/sec}} \quad = \quad \underline{4.1 \times 10^{-14} \text{ watts}}
\end{aligned}$$

Problem 5: Thermal Noise in Circuits III, Circuit Model for a Real Resistor

a)

$$\begin{aligned}
v_{\text{line}} &= \frac{R}{R+R} v_N(\omega) \\
&= \frac{1}{2} v_N(\omega)
\end{aligned}$$

$$\text{Power}\Big|_{\text{line}} = \left\langle \frac{v_{\text{line}}^2}{R} \right\rangle = \underline{\frac{1}{4R} < v_N^2(\omega) >}$$

b)

$$\frac{kT}{2\pi} = \frac{1}{4R} < v_N^2(\omega) > \quad \Rightarrow \quad < v_N^2(\omega) > = \underline{2RkT/\pi}$$

c) We are told that for the circuit shown

$$< v_C^2(\omega) >= \frac{1}{1+(RC\omega)^2} \; < v_0^2(\omega) >$$

when $v_0(\omega)$ is a random noise signal with zero mean. Identify R with the ideal resistor in the model of a real resistor in thermal equilibrium and $v_0(\omega)$ with the noise source voltage $v_N(\omega)$ in the model. Then

$$\begin{aligned}
< v_C^2(\omega) > &= \frac{1}{1+(RC\omega)^2} \; < v_N^2(\omega) > \\
&= \frac{2}{\pi}\, kT\, \frac{R}{1+(RC\omega)^2} \\
< v_C^2 > &= \int_0^\infty < v_C^2(\omega) > \, d\omega \\
&= \frac{2}{\pi}\, kT \int_0^\infty \frac{R}{1+(RC\omega)^2}\, d\omega \\
&= \frac{2}{\pi}\, \frac{kT}{C} \int_0^\infty \frac{1}{1+(RC\omega)^2}\, d(RC\omega) \\
&= \frac{2}{\pi}\, \frac{kT}{C} \underbrace{\int_0^\infty \frac{dx}{1+x^2}}_{\pi/2} = \underline{\frac{kT}{C}}
\end{aligned}$$

This is just the result we found using the canonical ensemble on the capacitor alone.

THE THERMAL NOISE IN A CIRCUIT CAN BE THOUGHT OF (AND QUANTITATIVELY MODELED) AS ARISING FROM THE DISSIPATIVE ELEMENTS IN THE SYSTEM.

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