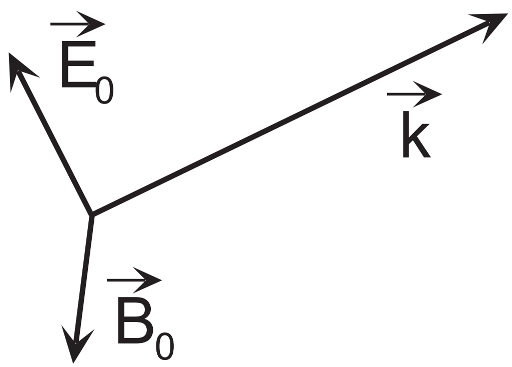
<u>Thermal Radiation</u> Radiation in thermal equilibrium with its surroundings



$$\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \omega = c|\vec{k}|$$

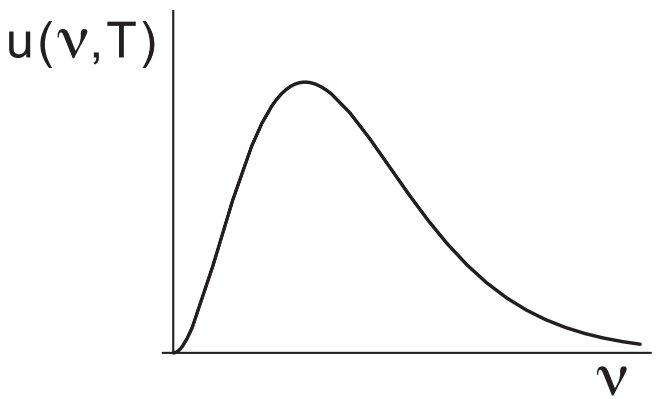
$$\vec{B} = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \vec{B}_0 = \vec{1}_k \times \vec{E}_0 \,/c$$

Time average energy density $\overline{u} = \frac{1}{2}\epsilon_0|\vec{E}_0|^2$

Time average energy flux $\vec{j}_E = (c\overline{u})\,\vec{1}_k$

Time average pressure ($\perp$ to $\vec{k}$) $P = \overline{u}$

Thermal radiation has a continuous distribution of frequencies.

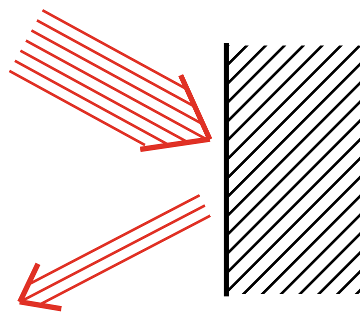


Peaks near $h\nu = 3k_BT$

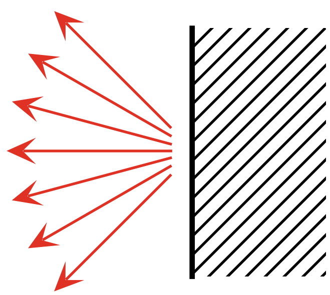
$(h/k_B \sim 5 \times 10^{-11}$ K-sec)

Spectral Region	ν (Hz)	T (K)	Thermal Rad.
Radio	10^6	1.7×10^{-5}	
Microwave	10^{10}	0.17	cosmic background
Infrared	10^{13}	1.7×10^2	room temp.
Visible	$\frac{1}{2} \times 10^{15}$	8.5×10^3	sun's surface
Ultraviolet	10^{16}	1.7×10^5	
X ray	10^{18}	1.7×10^7	black holes
γ ray	10^{21}	1.7×10^{10}	

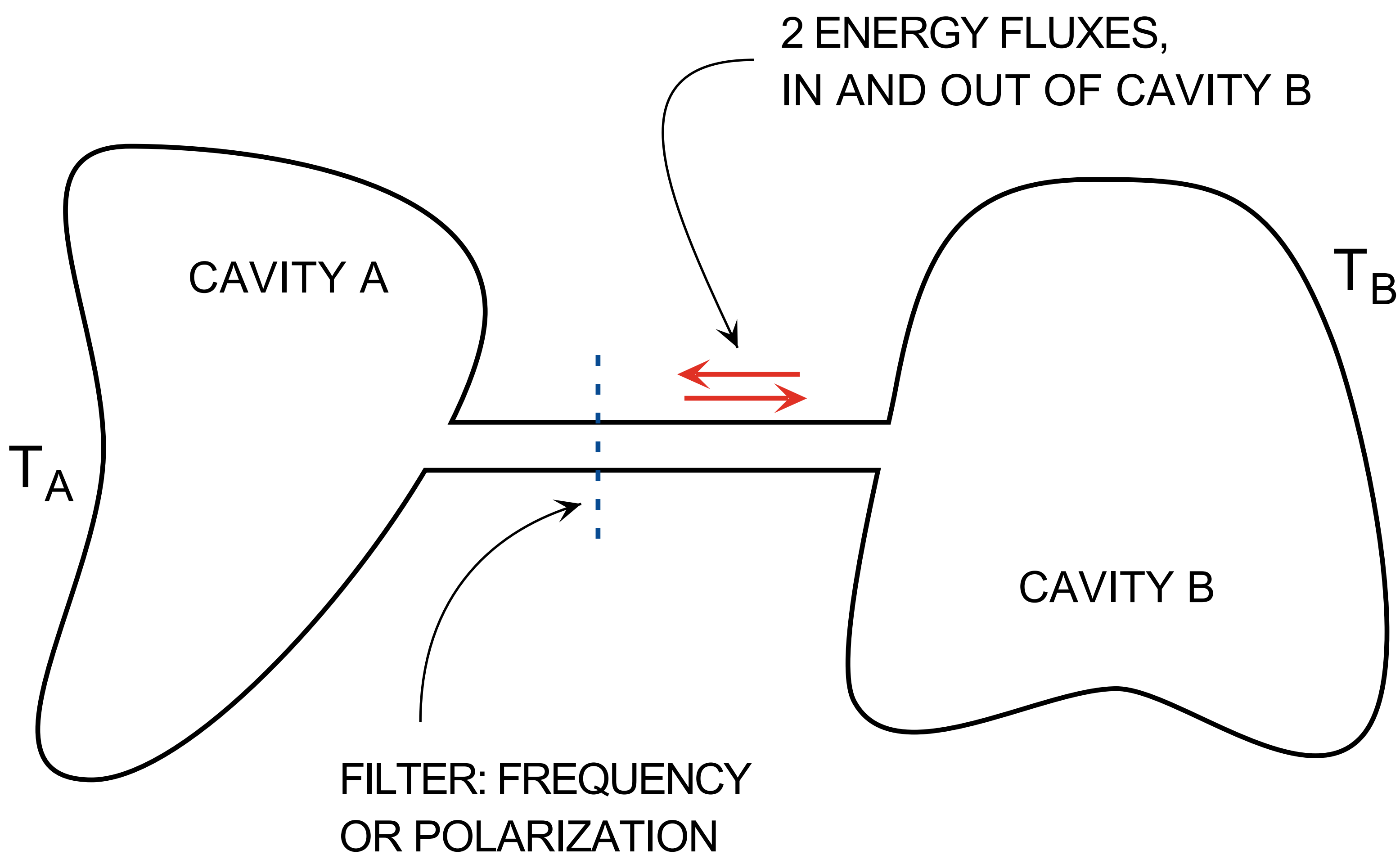
$$\text{ABSORPTIVITY } \alpha(\nu,T) \equiv \left\langle \frac{\text{ENERGY ABSORBED}}{\text{ENERGY INCIDENT}} \right\rangle_{\text{ISOTROPIC}}$$



$$\text{EMISSIVE POWER } e(\nu,T) \equiv \left\langle \frac{\text{ENERGY EMITTED}}{\text{AREA}} \right\rangle_{\text{ISOTROPIC}}$$



THERMAL RADIATION: PROPERTIES



ASSUME $T_A = T_B$ AND THERMAL EQUILIBRIUM

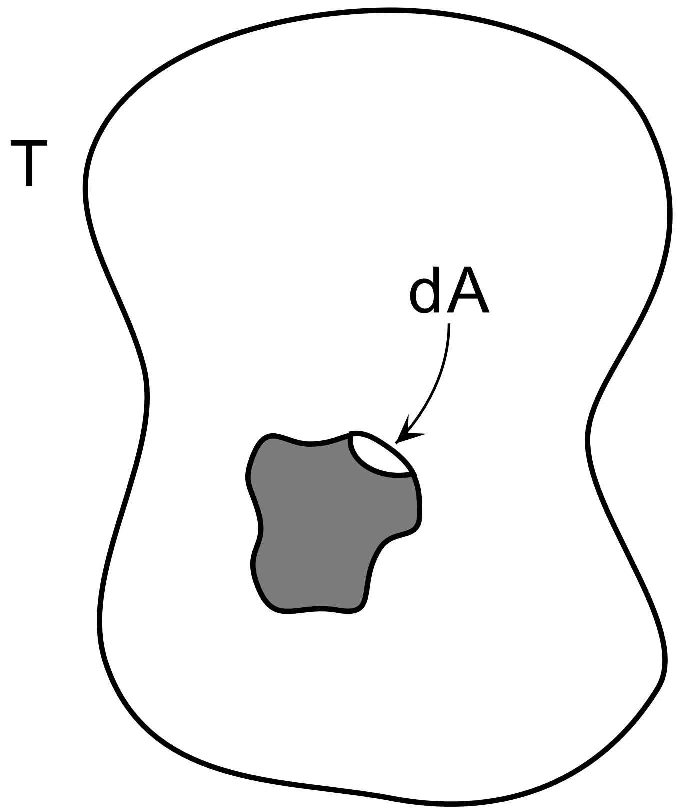
CONCLUSIONS:

- $u(\nu, T)$ is independent of shape and wall material

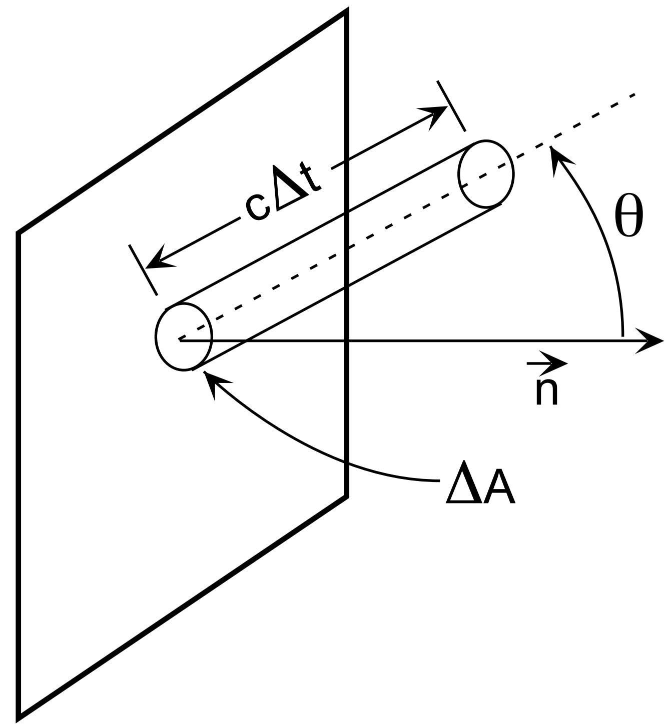
- $u(\nu, T)$ is isotropic

- $u(\nu, T)$ is unpolarized

CONSIDER AN OBJECT IN THE CAVITY, IN THERMAL EQUILIBRIUM



COMPUTE THE ENERGY FLUX



$$\begin{aligned}\Delta E &= \int (E \text{ in cylinder})\, p(\theta,\phi)\, d\theta\, d\phi \\ &= \int (u \;\; \Delta A \;\; \cos\theta \;\; c\Delta t) \left(\frac{\sin\theta}{2} \frac{1}{2\pi} \right) \, d\theta\, d\phi \\ &= c\, u\, \Delta A\, \Delta t \underbrace{\int_0^{\pi/2} \frac{\cos\theta \sin\theta}{2} d\theta}_{1/4} \underbrace{\int_0^{2\pi} \frac{1}{2\pi} d\phi}_{1}\end{aligned}$$

$\Rightarrow$ energy flux onto $dA = \underline{\frac{1}{4} c\, u(\nu, T)}$

Momentum Flux

Plane wave momentum density $\vec{p} = \frac{u}{c}\,\vec{1}_k$

$|\Delta p| = 2|p_\perp|$ since $\vec{p}_{\perp\,\text{in}} = -\vec{p}_{\perp\,\text{out}}$

$$
\begin{aligned}
|\Delta p|_\nu &= \int \left(\frac{2\cos\theta}{c}\right)(E \text{ in cylinder})\, p(\theta,\phi)\, d\theta\, d\phi \\
&= u(\nu,T)\,\Delta A\,\Delta t \underbrace{\int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta}_{1/3} \underbrace{\int_0^{2\pi} \frac{1}{2\pi}\, d\phi}_{1} \\
&= \tfrac{1}{3} u(\nu,T)\,\Delta A\,\Delta t
\end{aligned}
$$

$$
\Rightarrow P(T) = \underline{\tfrac{1}{3} \int_0^\infty u(\nu,T)\, d\nu}
$$

Apply detailed balance to the object in the cavity.

$$E_{\text{out}} = E_{\text{in}}$$

$$e\,dA = \alpha\left(\tfrac{1}{4}c\,u(\nu,T)\right)dA$$

$$\Rightarrow \quad \frac{e(\nu,T)}{\alpha(\nu,T)} = \tfrac{1}{4}c\,u(\nu,T)$$

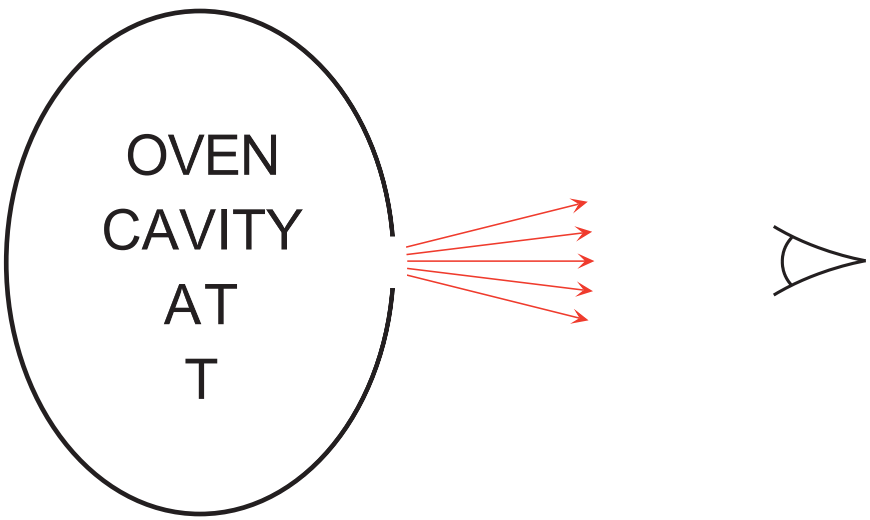
This ratio has a universal form for <u>all</u> materials.

The result is known as KIRCHOFF'S LAW.

Black Body Radiation

If $\alpha \equiv 1 \equiv$ "Black"

Then $e(\nu, T) = \frac{1}{4} c\, u(\nu, T)$



Measure $e(\nu, T)$

and obtain $u(\nu, T)$

Thermodynamic Approach

$$u(T) \equiv \int_0^\infty u(\nu, T)\, d\nu$$

Then

$$\begin{aligned} E(T,V) &= u(T)V \\ P(T,V) &= \tfrac{1}{3}u(T) \end{aligned}$$

This is enough to allow us to find $u(T)$.

$$\begin{aligned}
dE &= TdS - PdV \\
\left(\frac{\partial E}{\partial V}\right)_T &= T\left(\frac{\partial S}{\partial V}\right)_T - P = T\left(\frac{\partial P}{\partial T}\right)_V - P \\
&= \tfrac{1}{3}Tu'(T) - \tfrac{1}{3}u(T) \\
\text{also} &= u(T) \\
\Rightarrow \quad u'(T) &= \frac{4}{T}u(T) \\
u(T) &= AT^4
\end{aligned}$$

Emissive Power of a Black ($\alpha = 1$) Body

$$e(\nu, T) = \tfrac{1}{4} c\, u(\nu, T) \Rightarrow e(T) = \tfrac{1}{4} c\, u(T) = \tfrac{1}{4} A c T^4$$

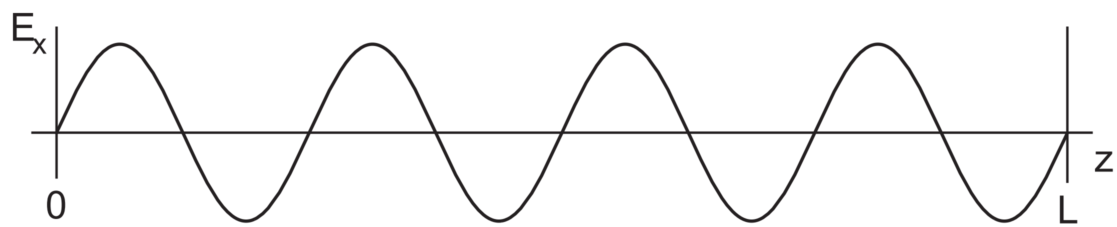
$$e(T) \equiv \sigma T^4$$

This is known as the STEFAN-BOLTZMANN LAW.

$$\sigma = 56.7 \times 10^{-9} \text{ watts}/m^2 K^4$$

Statistical Mechanical Approach

$\mathcal{H}$? Single normal mode (plane standing wave) in a rectangular conducting cavity.



$$\vec{E}_{0,0,n,\vec{1}_x}(\vec{r},t) = E(t)\sin(n\pi z/L)\vec{1}_x$$
$$\vec{B}_{0,0,n,\vec{1}_y}(\vec{r},t) = (n\pi c^2/L)^{-1}\dot{E}(t)\cos(n\pi z/L)\vec{1}_y$$

Energy density $= \frac{1}{2}\epsilon_0 \vec{E}\cdot\vec{E} + \frac{1}{2}\frac{1}{\mu_0}\vec{B}\cdot\vec{B}$ [no $\vec{r}$ or t average]

$$\mathcal{H} = \frac{V}{2}\left[\tfrac{1}{2}\epsilon_0 E^2(t) + \tfrac{1}{2}\frac{1}{\mu_0}(n\pi c^2/L)^{-2}\dot{E}^2(t)\right]$$

$$= \frac{V}{2}\frac{\epsilon_0}{2}\left[\, E^2(t) + (n\pi c/L)^{-2}\,\dot{E}^2(t)\right]$$

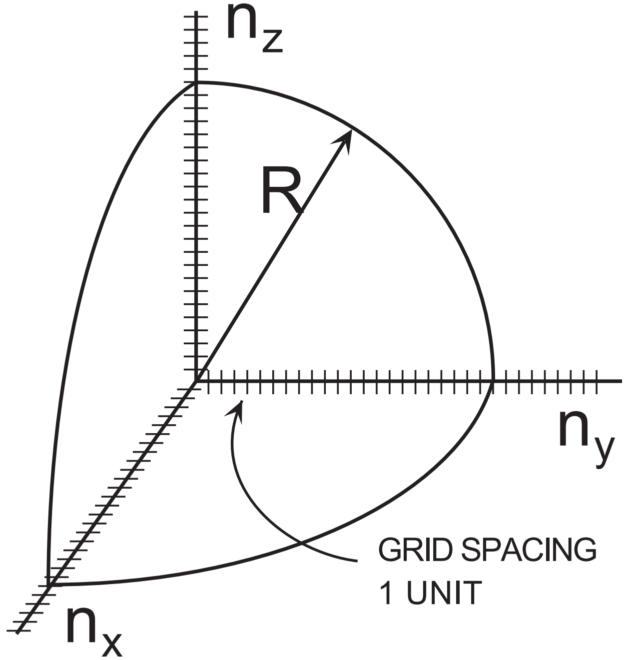
$\Rightarrow$ Each mode corresponds to a harmonic oscillator.

Count the modes.

$$\vec{E}_{n_x,n_y,n_z} = |E|\vec{\epsilon}_j \sin(n_x\pi x/L) \sin(n_y\pi y/L) \sin(n_z\pi z/L)e^{i\omega t}$$

The unit polarization vector $\vec{\epsilon}_j$ has 2 possible orthogonal directions and $n_i = 1, 2, 3 \cdots$.

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2\,\vec{\nabla}^2 E = 0 \quad \Rightarrow \quad \omega^2 = \left(\frac{\pi c}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2)$$



If the radian frequency $< \omega$

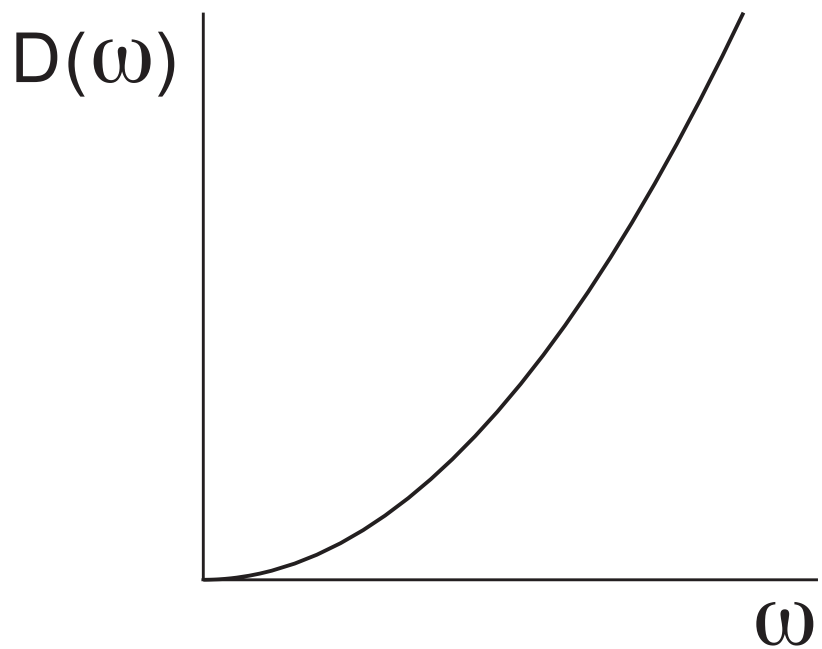
$$R = \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{L}{\pi c}\,\omega$$

$\#$ modes (freq. $< \omega$)

$$= 2 \times \frac{1}{8} \times \frac{4}{3}\pi R^3$$

$$= \frac{\pi}{3}\left(\frac{L}{\pi c}\right)^3 \omega^3$$

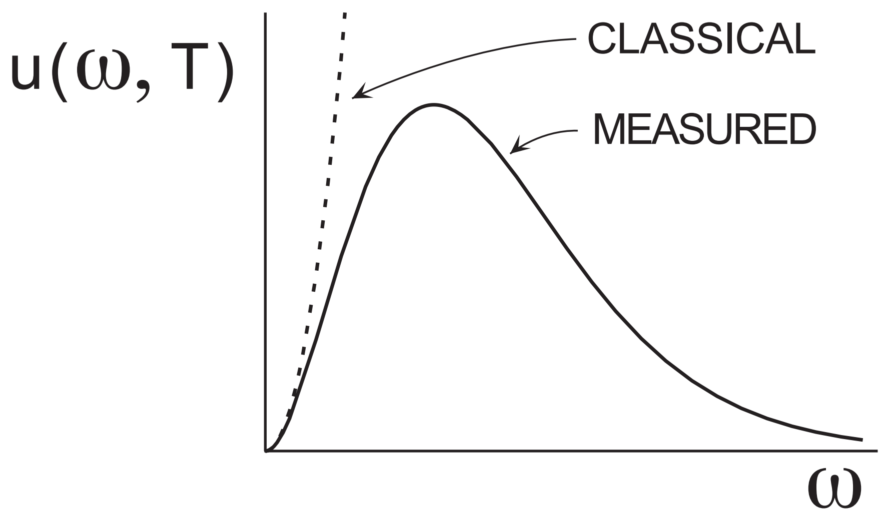
$$D(\omega) = \frac{d\#}{d\omega} = \pi \left(\frac{L}{\pi c}\right)^3 \omega^2 = \frac{V}{\pi^2 c^3}\,\omega^2$$



Classical Statistical Mechanics

$$< \epsilon(\omega) >= k_B T \quad \Rightarrow \quad u(\omega, T) =< \epsilon(\omega) > \frac{D(\omega)}{V} = \frac{k_B T}{\pi^2 c^3} \omega^2$$

$$u(T) = \int_0^\infty u(\omega, T)\, d\omega = \infty$$



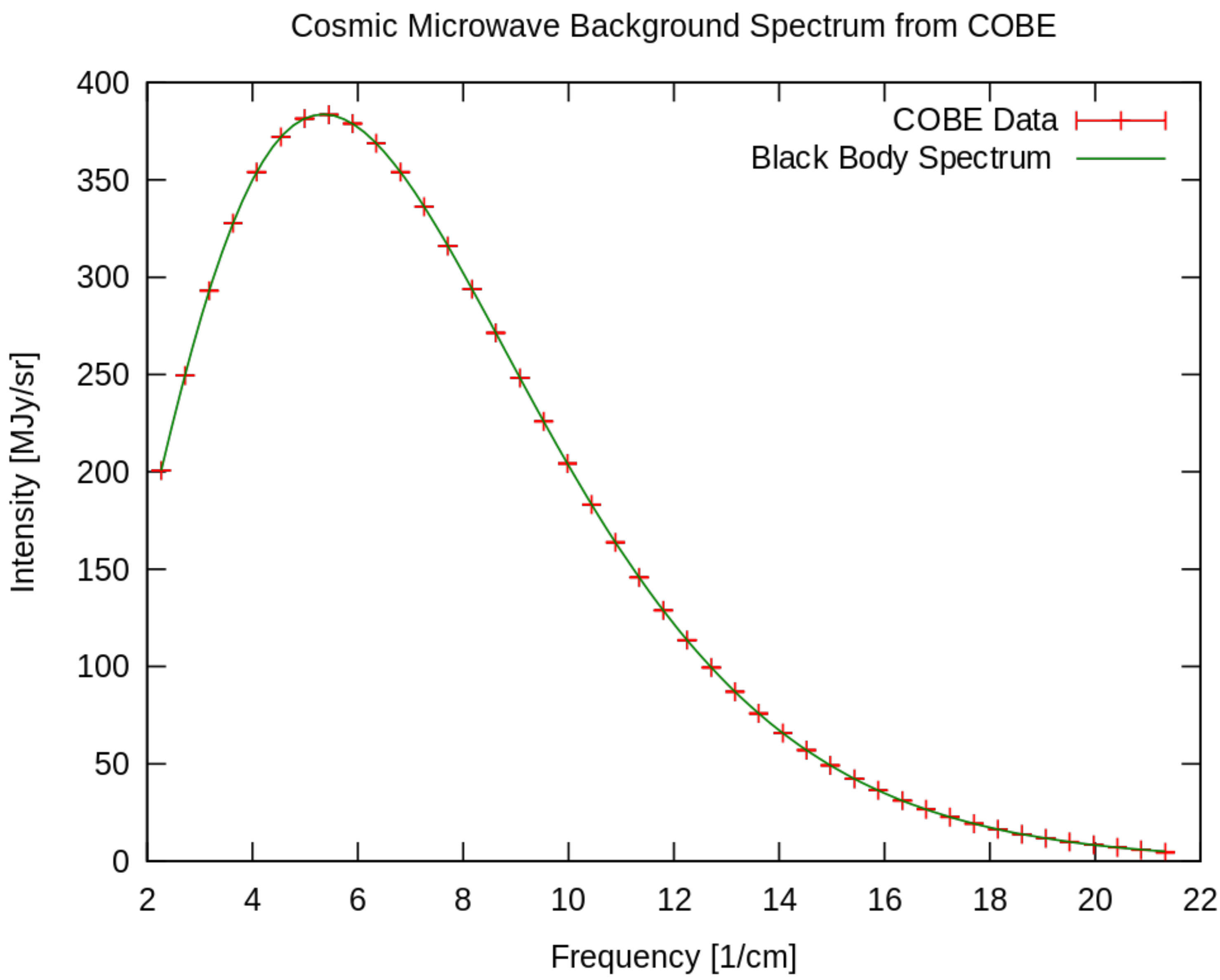
Quantum Statistical Mechanics

$$< \epsilon(\omega) > = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} + \hbar\omega/2$$

$$u(\omega, T) = < \epsilon(\omega) > \frac{D(\omega)}{V} = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1} + \text{z. p. term}$$

To find the location of the maximum, set $\frac{du(\omega, T)}{d\omega} = 0$.

The maximum occurs at $\hbar\omega/kT \approx 2.82$.


Cosmic Microwave Background Spectrum from COBE
COBE Data
Black Body Spectrum
Intensity [MJy/sr]
Frequency [1/cm]
0
50
100
150
200
250
300
350
400
2
4
6
8
10
12
14
16
18
20
22

$$Z = \prod_{\text{states } i} Z_i \qquad\qquad Z_i = e^{-\hbar\omega/2kT}(1 - e^{-\hbar\omega/kT})^{-1}$$

The first factor in the expression for Z_i comes from the zero-point energy.

$$F(V,T) = -kT \ln Z = -kT \sum_{\text{states } i} \ln Z_i$$

$$= -kT \int_0^\infty D(\omega)\,[\ln Z_i]\; d\omega$$

$$
\begin{aligned}
F(V,T) &= -kT \int_0^\infty D(\omega)\left[-\ln(1-e^{-\hbar\omega/kT})\right] d\omega + \cdots \\
&= \frac{kTV}{\pi^2 c^3} \int_0^\infty \omega^2 \ln(1-e^{-\hbar\omega/kT})\, d\omega \\
&= \frac{V}{\pi^2 c^3 \hbar^3} (kT)^4 \underbrace{\int_0^\infty x^2 \ln(1-e^{-x})\, dx}_{-\frac{\pi^4}{45}} \\
&= \underline{-\frac{1}{45}\frac{\pi^2}{c^3\hbar^3}(kT)^4 V}
\end{aligned}
$$

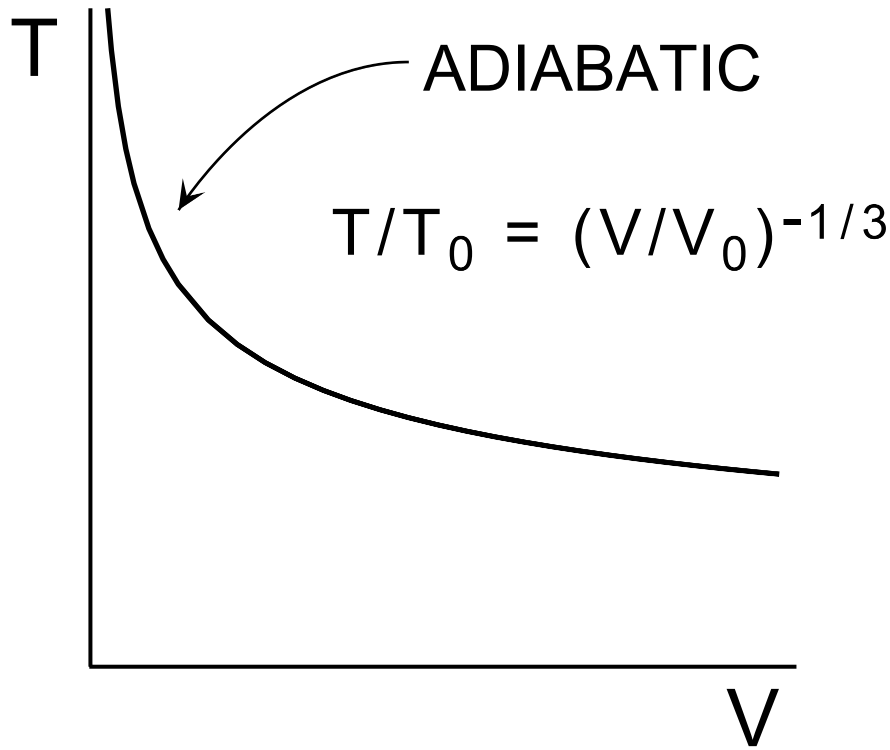
$$P = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{1}{45}\frac{\pi^2}{(c\hbar)^3}(kT)^4$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4}{45}\frac{\pi^2}{(c\hbar)^3}k^4T^3V$$

$$E = F + TS = \left(-\frac{1}{45} + \frac{4}{45}\right)(\cdots) = \frac{1}{15}\frac{\pi^2}{(c\hbar)^3}(kT)^4V$$

Note: $P = \frac{1}{3}E/V = \frac{1}{3}u(T)$ independent of V.

NOTE: THE ADIABATIC PATH IS T^3V=CONSTANT



8.044 Statistical Physics I
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