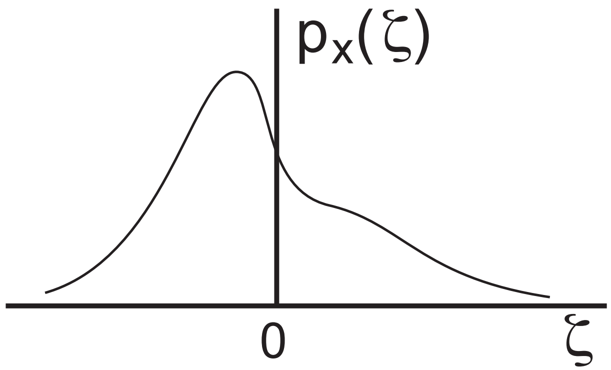
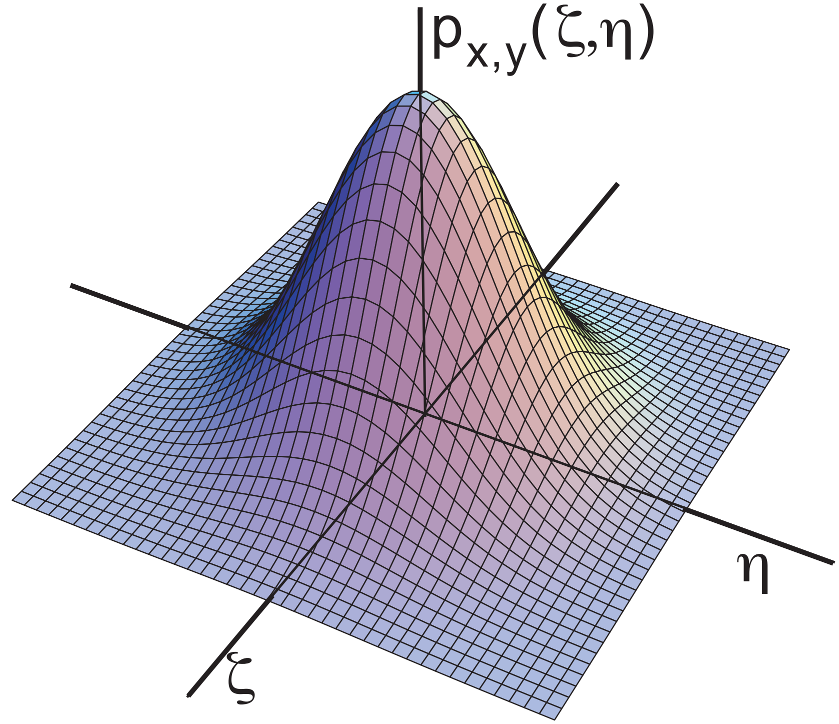


p_x(ζ)
0
ζ



p_x,y(ζ,η)
η
ζ

$$p_{x,y}(\zeta,\eta)d\zeta d\eta \equiv \text{PROB}(\zeta < x \leq \zeta{+}d\zeta \text{ and } \eta < y \leq \eta{+}d\eta)$$

$$\begin{aligned} P_{x,y}(\zeta,\eta) &\equiv \text{PROB}(x \leq \zeta \text{ and } y \leq \eta) \\ &= \int_{-\infty}^{\zeta}\int_{-\infty}^{\eta} p_{x,y}(\zeta',\eta')d\zeta' d\eta' \end{aligned}$$

$$p_{x,y}(\zeta,\eta) = \frac{\partial}{\partial\zeta}\frac{\partial}{\partial\eta}P_{x,y}(\zeta,\eta)$$

$$\langle f(x,y)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\zeta,\eta)p_{x,y}(\zeta,\eta)d\zeta d\eta$$

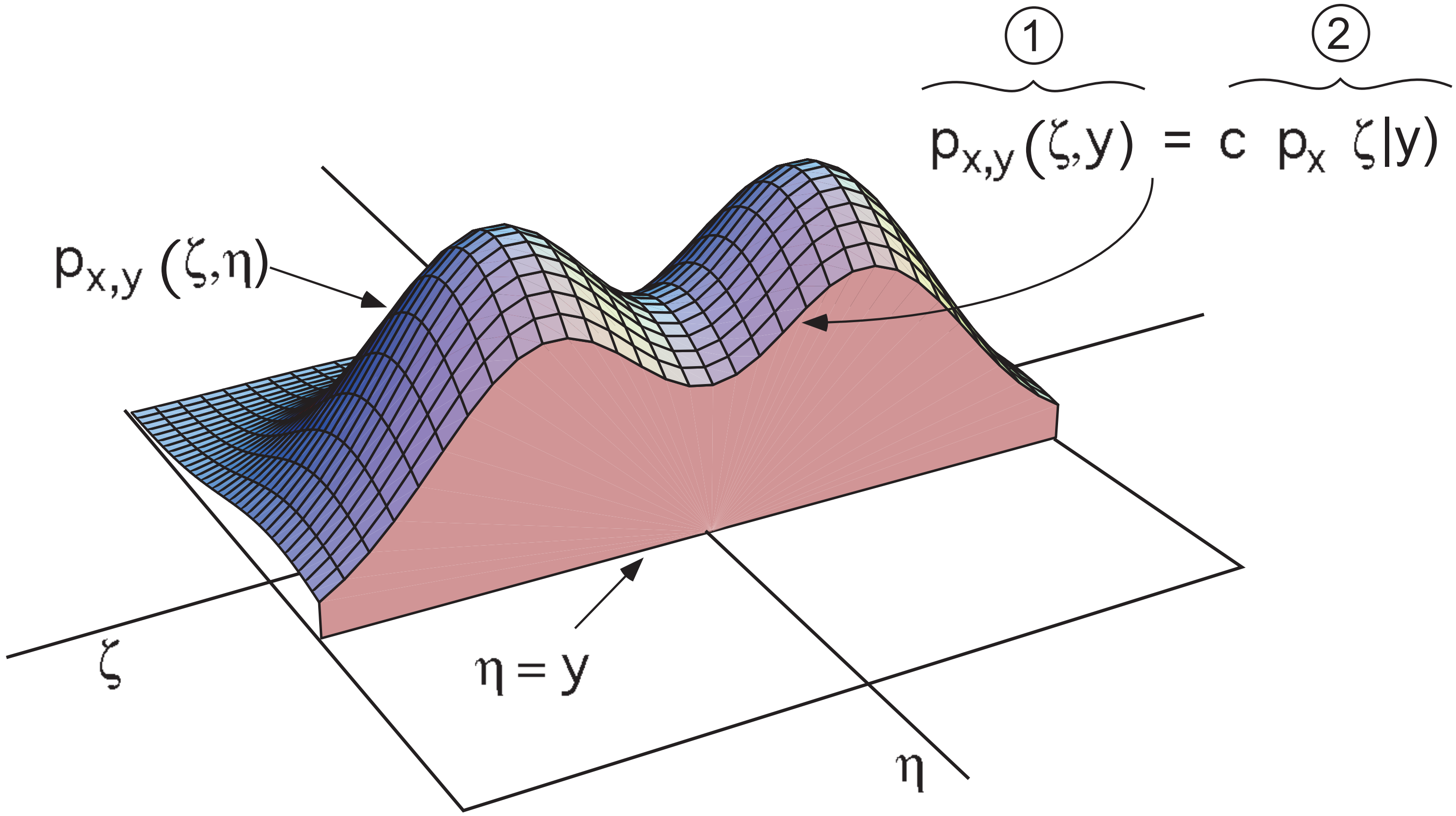
NEW CONCEPTS:

Reduction to a single variable

$$p_x(\zeta) = \int_{-\infty}^{\infty} p_{x,y}(\zeta,\eta)d\eta$$

Conditional probability density

$$p_x(\zeta|y)d\zeta \equiv \text{prob.}(\zeta < x \leq \zeta + d\zeta \text{ given that } \eta = y)$$


1
2
$p_{x,y}(\zeta,y) = c\ p_x\ \zeta|y)$
$p_{x,y}(\zeta,\eta)$
ζ
$\eta = y$
η

$$① \qquad \int_{-\infty}^{\infty} p_{x,y}(\zeta, y)\, d\zeta = p_y(\eta = y)$$

$$② \qquad c \underbrace{\int_{-\infty}^{\infty} p_x(\zeta | y)\, d\zeta}_{1} = c$$

$$\Rightarrow \quad c = p_y(\eta = y)$$

$$p_x(\zeta | y) = \frac{p_{x,y}(\zeta, y)}{p_y(\eta = y)} \quad \text{or} \quad p(x|y) = \frac{p(x, y)}{p(y)}$$

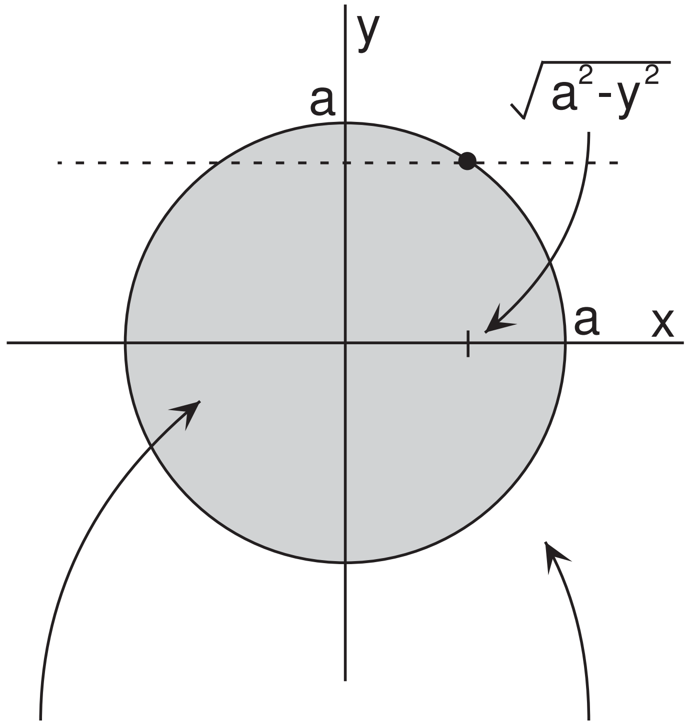
Bayes' Theorem or "fundamental law of conditional probability"

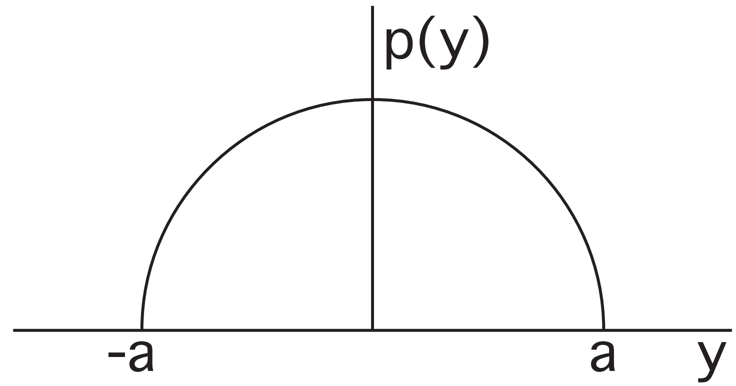
$$p(x,y) = p(x|y)p(y)$$

$$p(x,y) = p(y|x)p(x)$$

x and y are statistically independent if $p(x,y) = p(x)p(y)$

$$\Rightarrow \underbrace{p(x|y)}_{\text{conditioned}} = \underbrace{p(x)}_{\text{unconditioned}} \qquad\qquad p(y|x) = p(y)$$



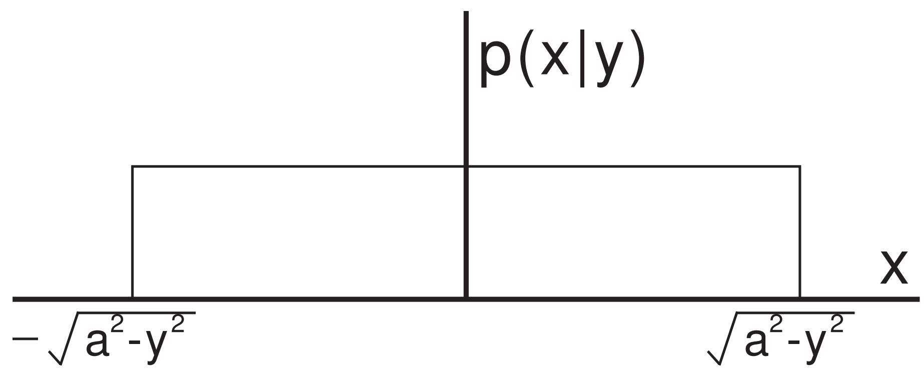


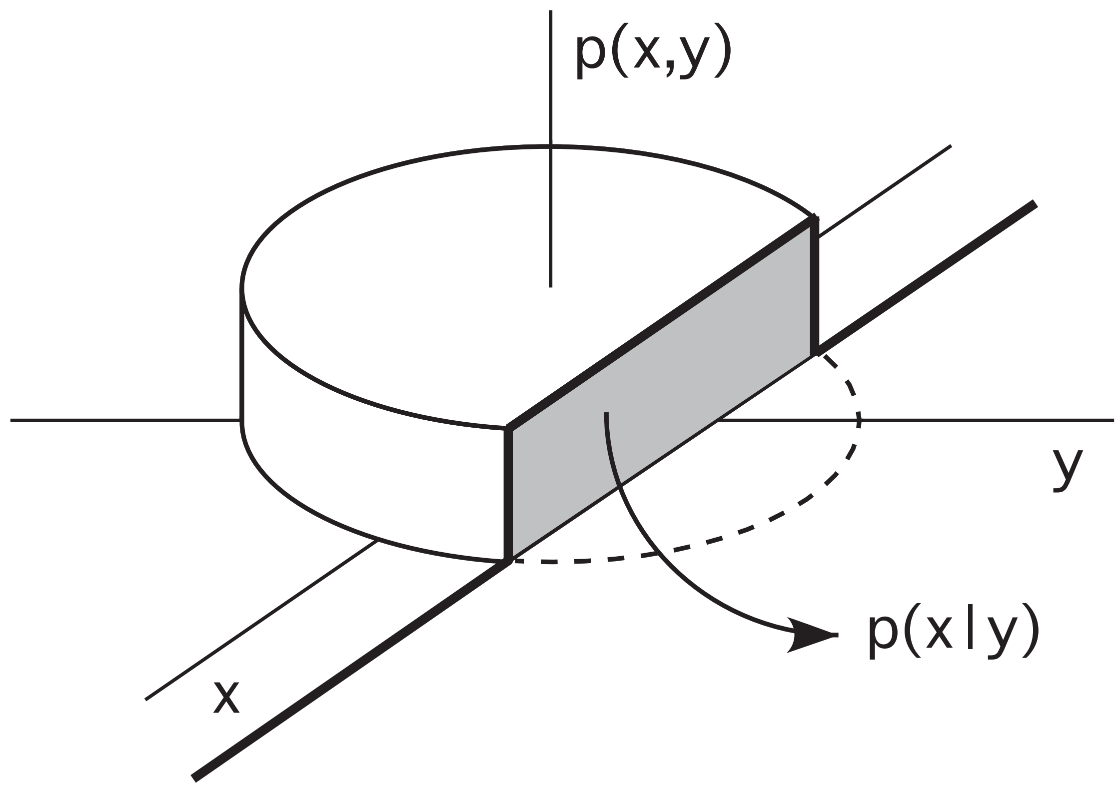
$$p(x,y) = 1/\pi a^2 \qquad p(x,y) = 0$$

$$p(y) = \frac{2}{\pi a}\sqrt{1-(\frac{y}{a})^2}$$

$$p(x|y) = \frac{1}{2\sqrt{a^2 - y^2}} \qquad |x| \leq \sqrt{a^2 - y^2}$$

$$= 0 \qquad |x| > \sqrt{a^2 - y^2}$$



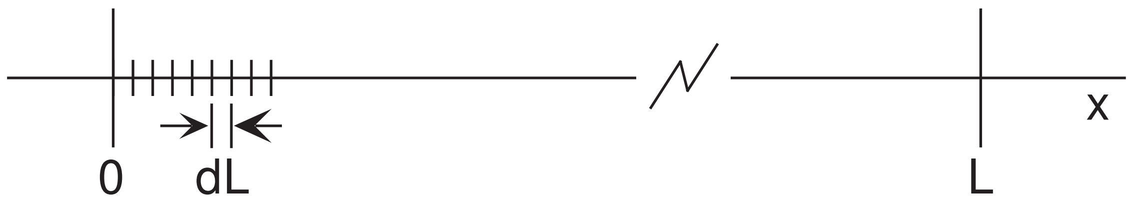


8.044 L2B9

Derivation of Poisson density p(n;L)

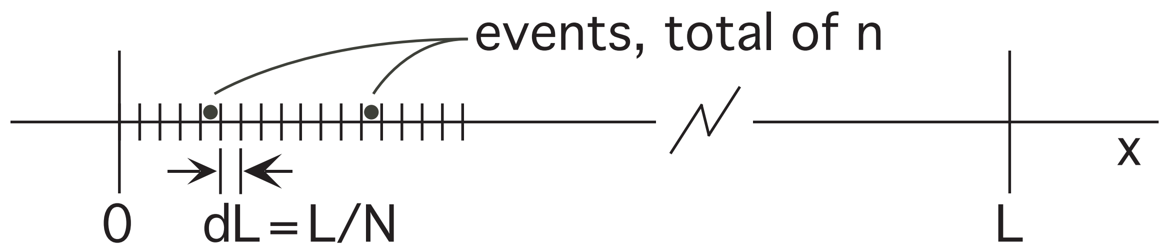
Given: $p(n = 1; \Delta x) \to r\Delta x$ as $\Delta x \to 0$

Start by finding $p(n = 0; L)$



$dL \to 0 \Rightarrow p(0) \gg p(1) \gg p(n > 1)$

$\Rightarrow p(n = 0, dL) \approx 1 - p(1) = 1 - r\,dL$



n is finite while $N \to \infty$

For a <u>given</u> set of n events in the N boxes

$$\text{PROB} = \left(\frac{rL}{N}\right)^n \left(1 - \left(\frac{rL}{N}\right)\right)^{(N-n)}$$

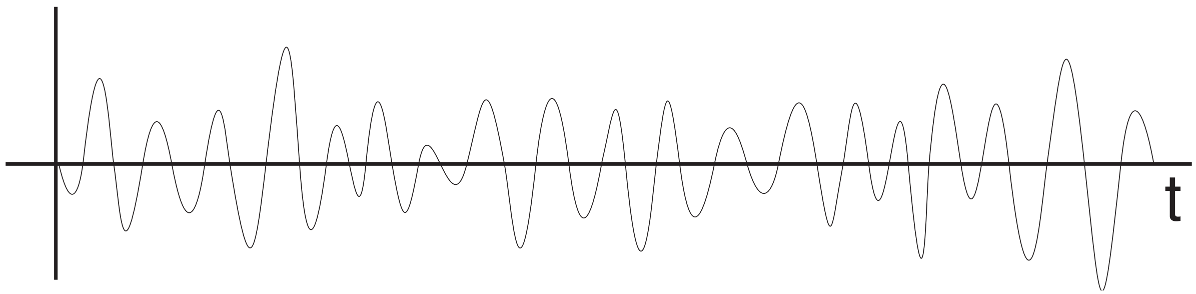
For $p(n;L)$ we must consider all possible ways the n events can be distributed in the N boxes, and the events are all identical.

$$p(n;L) = \lim_{N\to\infty} \frac{N!}{n!(N-n)!}\left(\frac{rL}{N}\right)^n \left(1-\left(\frac{rL}{N}\right)\right)^{(N-n)}$$

$$= \frac{1}{n!} \underbrace{\frac{N!}{(N-n)!}}_{\to N^n} \frac{1}{N^n} (rL)^n \underbrace{\left(1-\left(\frac{rL}{N}\right)\right)^{(N-n)}}_{\to e^{-rL}}$$

$$= \frac{1}{n!}(rL)^n e^{-rL} = \frac{1}{n!} < n >^n e^{-<n>}$$

Example Jointly Gaussian random variables



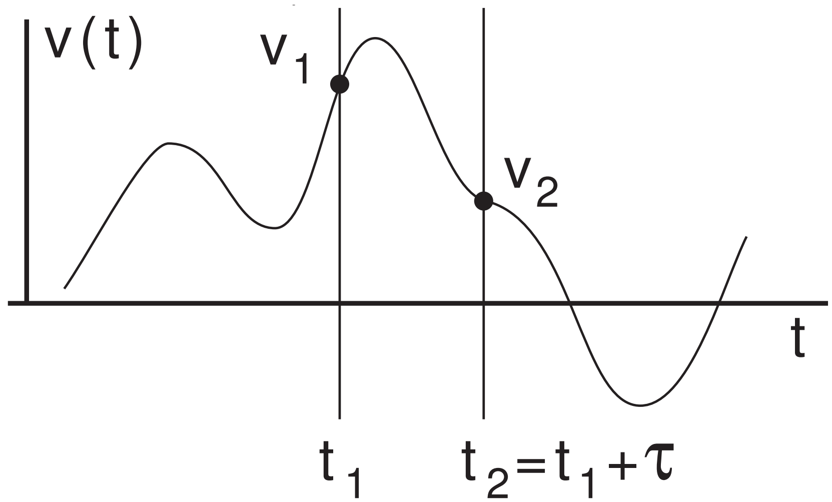
Random processes:

Noise voltage or current $v(t), i(t)$

Thermodynamic variable $P(t), T(t), \rho(t)$

Thermal radiation $E(t), B(t)$

Seismic background signal $s(t)$



$$p(v_1, v_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{v_1^2 - 2\rho v_1 v_2 + v_2^2}{2\sigma^2(1-\rho^2)}\right]$$

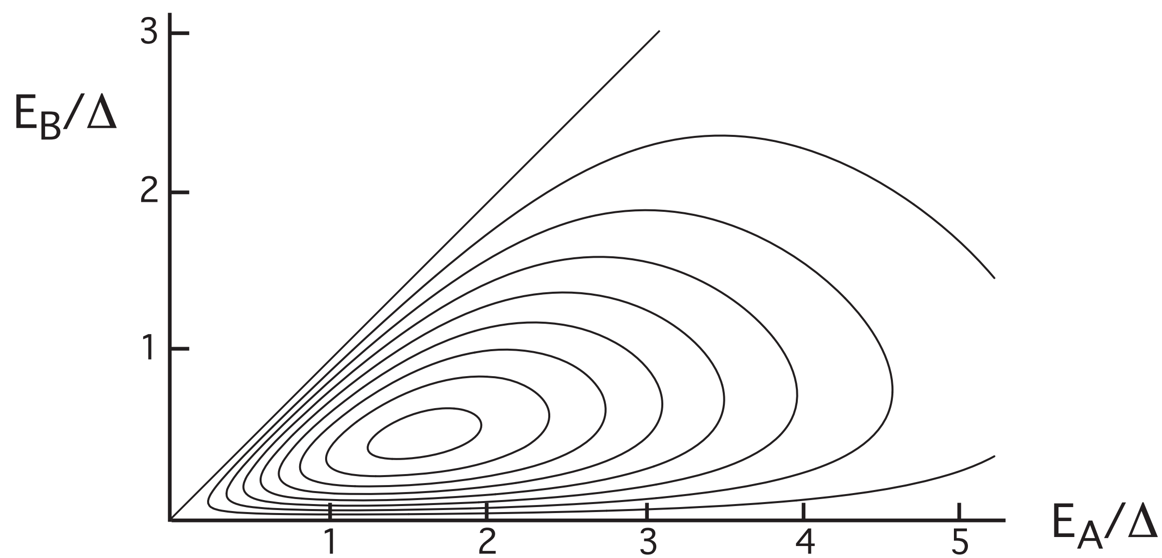
σ is constant

$\rho = \rho(\tau) \rightarrow 1$ as $\tau \rightarrow 0$

$\rightarrow 0$ as $\tau \rightarrow \infty$

Example (from an old exam)

Problem 2 (40 points) Collision Products



A certain collision process in high energy physics produces a number of biproducts. When the biproducts include a pair of elementary particles A and B the energies of those particles, E_A and E_B, are distributed according to the joint probability density

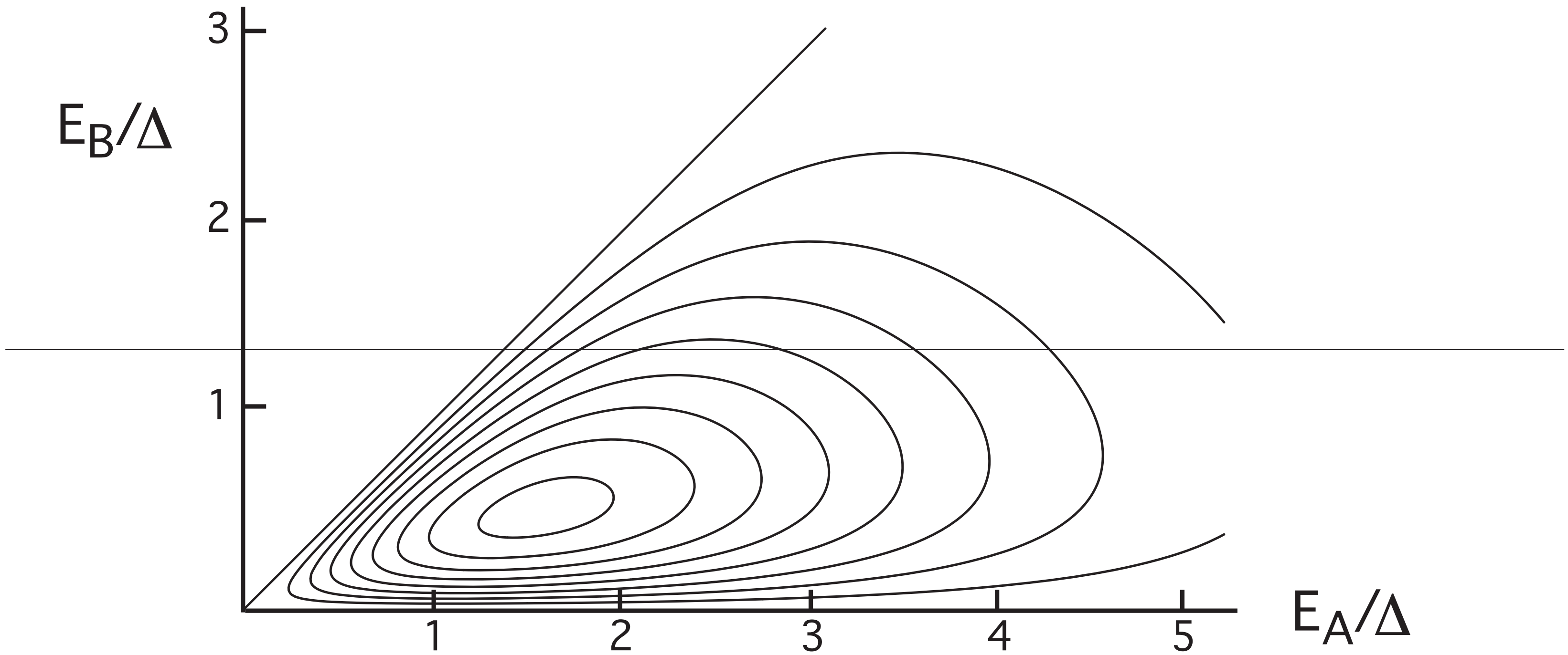
$$p(E_A, E_B) = \frac{4E_B(E_A - E_B)}{\Delta^4} \exp[-(E_A + E_B)/\Delta] \qquad \text{for } E_A > 0 \text{ and } E_A > E_B > 0$$
$$= 0 \qquad \text{elsewhere}$$

Δ is a parameter with the units of energy. A contour plot of $p(E_A, E_B)$ is shown above. Note that the energy E_A is always positive and greater than the energy E_B.

a) Find $p(E_B)$. Sketch the result.

b) Find the conditional probability density $p(E_A \mid E_B)$. Sketch the result.

c) Are E_A and E_B statistically independent? Explain your reasoning.



E_B/Δ
3
2
1
E_A/Δ
1
2
3
4
5

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8.044 Statistical Physics I
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