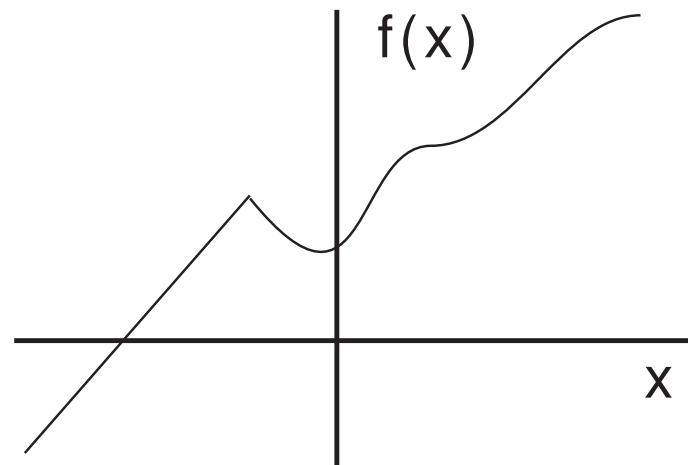


Functions of a random variable

Given: $p_x(\zeta)$ and $f(x)$

Find: $p_f(\eta)$



- A. Sketch $f(x)$. Find where $f(x) < \eta$

- B. Integrate to find $P_f(\eta)$.

- C. Differentiate to find $p_f(\eta)$.

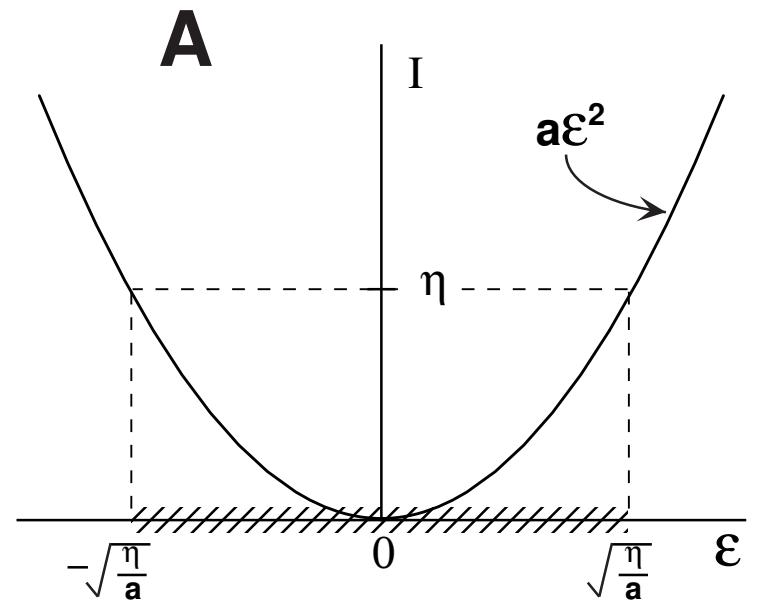
Example Intensity of light

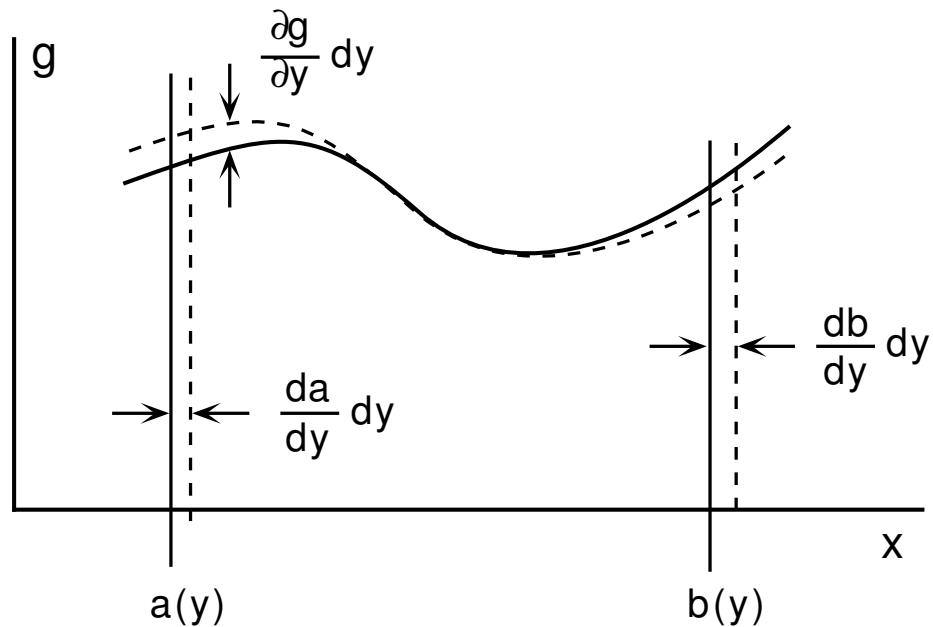
$$I = a\epsilon^2$$

$$p(\epsilon) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp[-\epsilon^2/2\sigma^2]$$

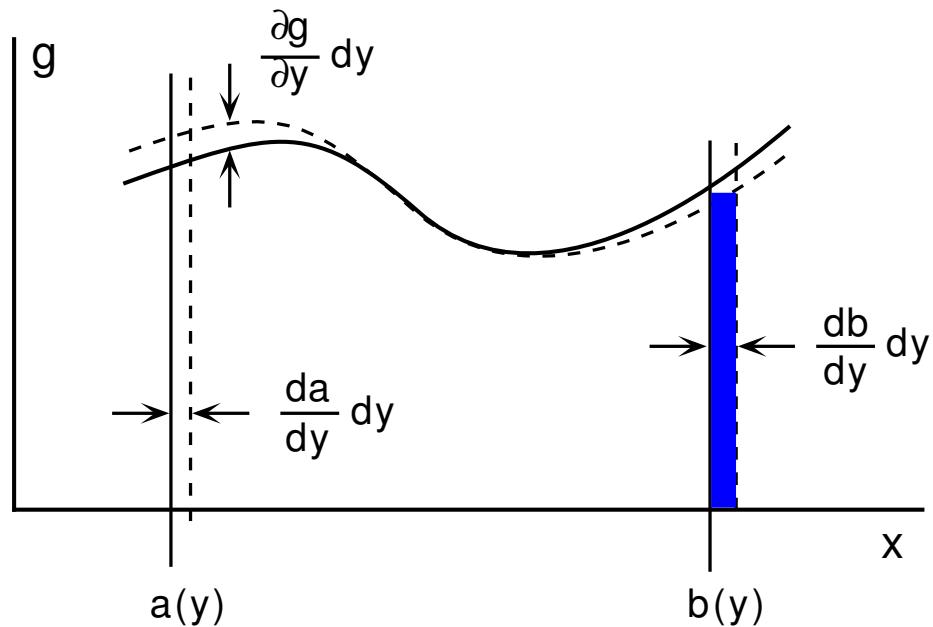
B

$$P_I(\eta) = \int_{-\sqrt{\eta/a}}^{\sqrt{\eta/a}} p_\epsilon(\zeta) d\zeta$$



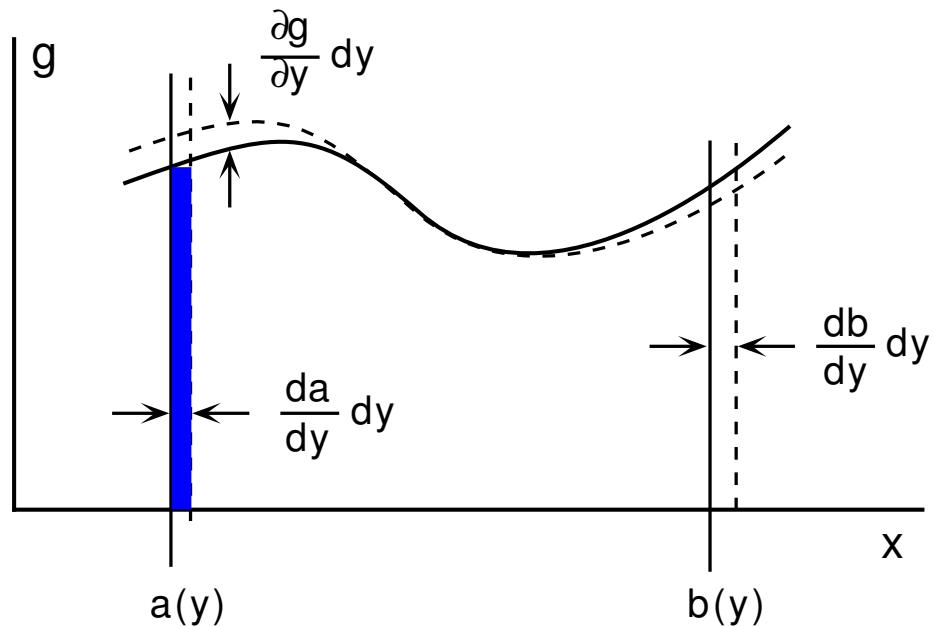


$$g(y, x = b(y)) \frac{db(y)}{dy} - g(y, x = a(y)) \frac{da(y)}{dy} + \int_{a(y)}^{b(y)} \frac{\partial g(y, x)}{\partial y} dx$$



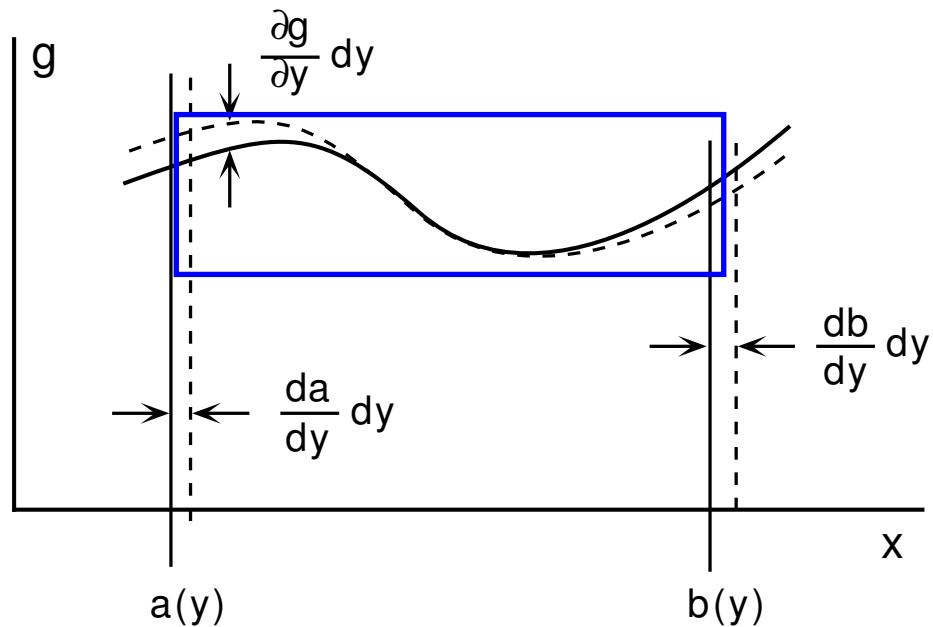
$$\frac{d}{dy} \int_{a(y)}^{b(y)} g(y, x) dx =$$

$$g(y, x = b(y)) \frac{db(y)}{dy} - g(y, x = a(y)) \frac{da(y)}{dy} + \int_{a(y)}^{b(y)} \frac{\partial g(y, x)}{\partial y} dx$$



$$\frac{d}{dy} \int_{a(y)}^{b(y)} g(y, x) dx =$$

$$g(y, x = b(y)) \frac{db(y)}{dy} - g(y, x = a(y)) \frac{da(y)}{dy} + \int_{a(y)}^{b(y)} \frac{\partial g(y, x)}{\partial y} dx$$



$$\frac{d}{dy} \int_{a(y)}^{b(y)} g(y, x) dx =$$

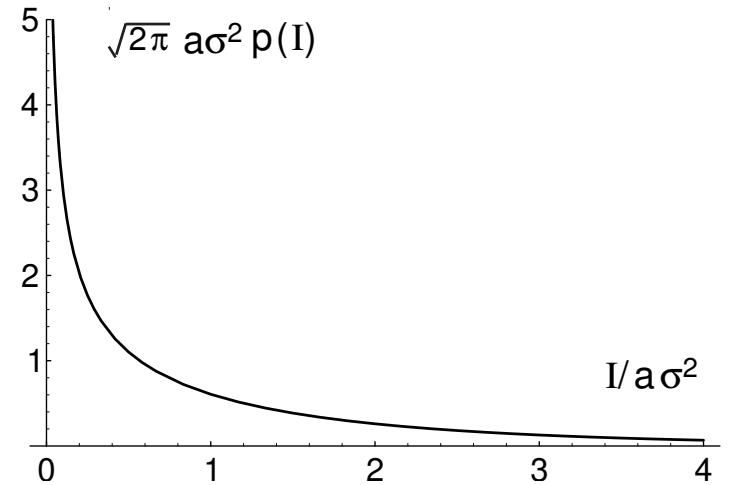
$$g(y, x = b(y)) \frac{db(y)}{dy} - g(y, x = a(y)) \frac{da(y)}{dy} + \boxed{\int_{a(y)}^{b(y)} \frac{\partial g(y, x)}{\partial y} dx}$$

C In general

$$\begin{aligned} p_I(\eta) &= \frac{1}{2} \frac{1}{\sqrt{\eta a}} p_{\mathcal{E}}(\sqrt{\eta/a}) - \left(-\frac{1}{2} \frac{1}{\sqrt{\eta a}}\right) p_{\mathcal{E}}(-\sqrt{\eta/a}) \\ &= \frac{1}{2} \frac{1}{\sqrt{\eta a}} [p_{\mathcal{E}}(\sqrt{\eta/a}) + p_{\mathcal{E}}(-\sqrt{\eta/a})] \end{aligned}$$

In our particular case

$$p(I) = \frac{1}{\sqrt{2\pi a\sigma^2 I}} \exp\left[-\frac{I}{2a\sigma^2}\right]$$

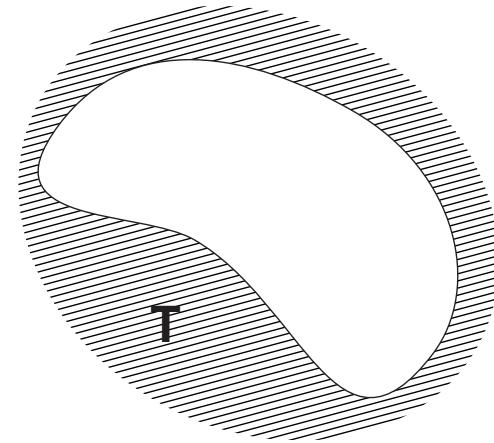
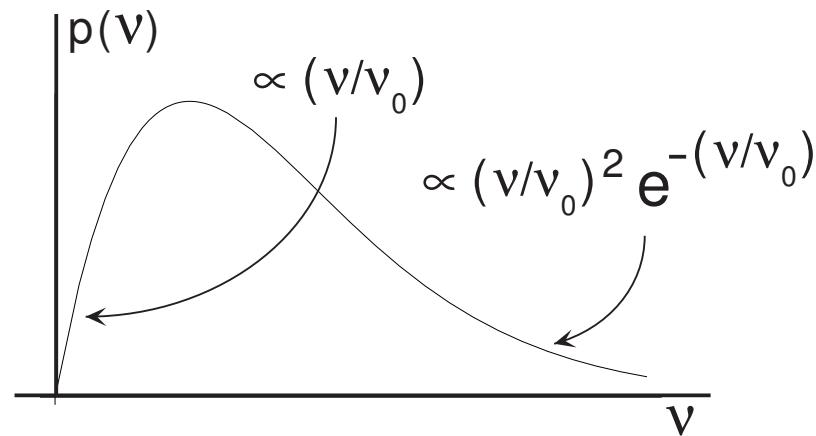


8.044 L3B5

Example Black Body Radiation

$$p(\nu) = \frac{1}{\underbrace{2\zeta(3)}_{1/2.404}} \frac{1}{\nu_0} \frac{(\nu/\nu_0)^2}{\exp[\nu/\nu_0] - 1}$$

$$\nu_0 = kT/h$$

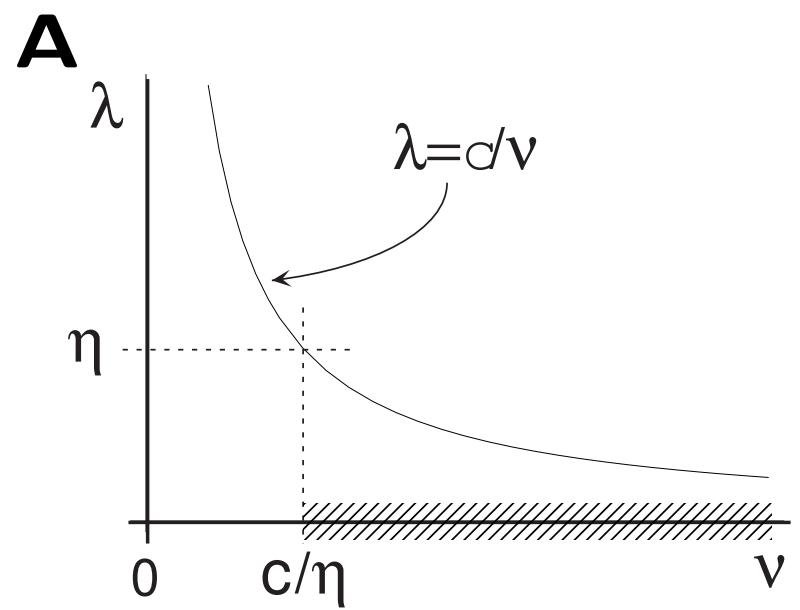


Given $\lambda = c/\nu$ and $p(\nu)$

Find $p(\lambda)$

B

$$P_\lambda(\eta) = \int_{c/\eta}^{\infty} p_\nu(\zeta) d\zeta$$



8.044 L3B7

C

In general

$$p_\lambda(\eta) = -(-c/\eta^2) p_\nu(c/\eta)$$

In our case

$$p_\lambda(\eta) = \frac{c}{\eta^2} \frac{1}{2.404} \frac{1}{\nu_0} \frac{(c/\eta\nu_0)^2}{\exp[(c/\eta\nu_0)] - 1}$$

Let $\lambda_0 \equiv c/\nu_0$, then

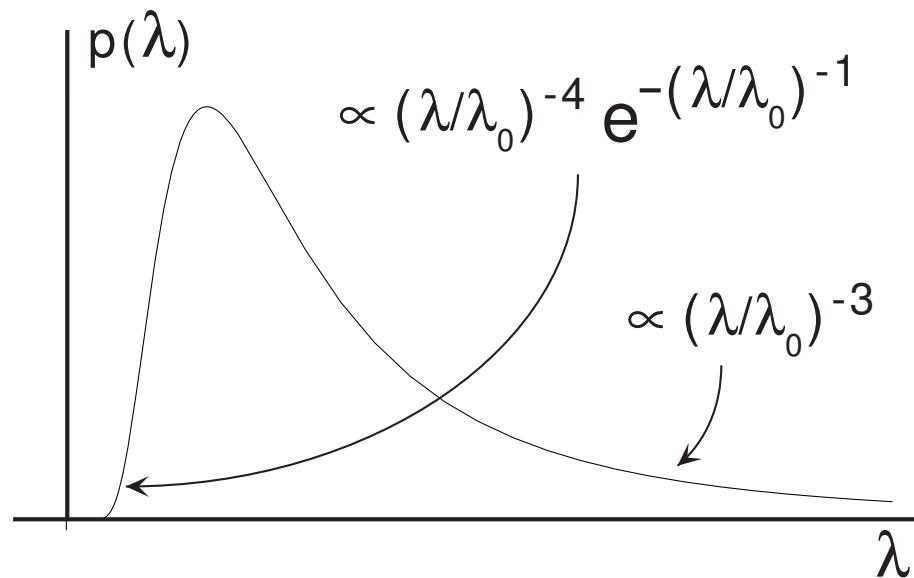
$$p_\lambda(\eta) = \frac{1}{2.404} \frac{1}{\lambda_0} \left(\frac{\eta}{\lambda_0} \right)^{-4} \frac{1}{\exp[(\eta/\lambda_0)^{-1}] - 1}$$

As $(\lambda/\lambda_0) \rightarrow 0$

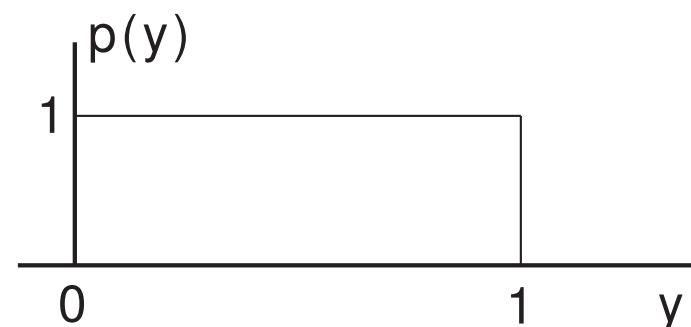
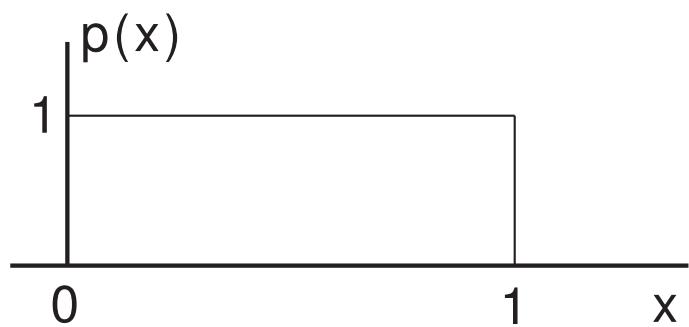
$$\frac{1}{\exp[(\lambda/\lambda_0)^{-1}] - 1} \rightarrow e^{-(\lambda/\lambda_0)^{-1}}$$

As $(\lambda/\lambda_0) \rightarrow \infty$

$$\frac{1}{\exp[(\lambda/\lambda_0)^{-1}] - 1} \rightarrow \frac{1}{(1 + (\lambda/\lambda_0)^{-1} - 1)} \rightarrow (\lambda/\lambda_0)$$



Example Random number generator for programmers



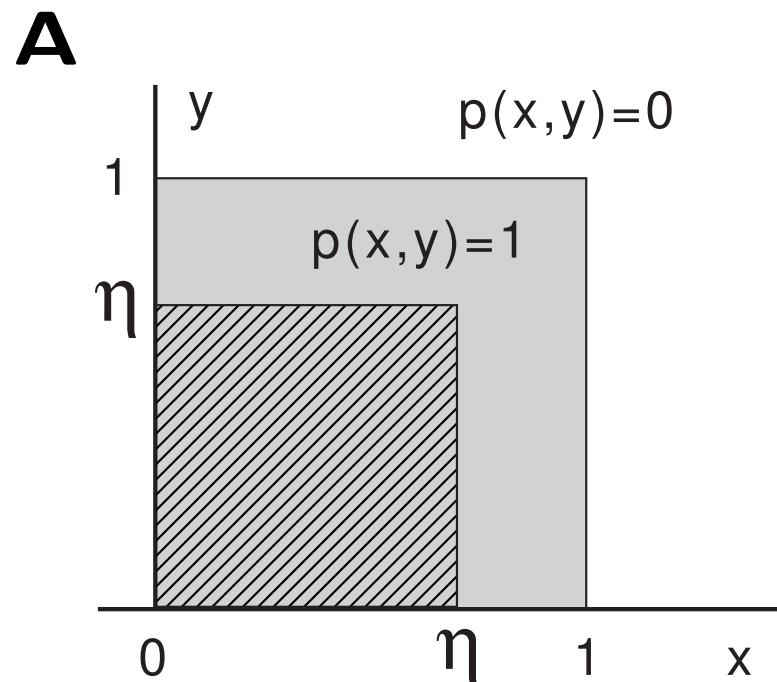
x and y are statistically independent

$z \equiv \text{MAX}(x, y)$ Find $p(z)$

$$p(x, y) = p(x) p(y)$$

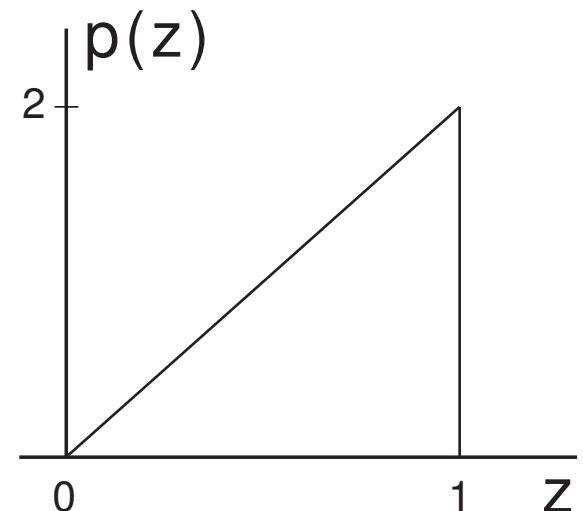
Where is $\text{MAX}(x, y) = \eta$?

Where is $\text{MAX}(x, y) < \eta$?



B $P_z(\eta) = \eta^2$

C $p_z(\eta) = 2\eta \quad 0 \leq \eta \leq 1$



$$\begin{aligned} \langle z \rangle &= \int_0^1 2\eta^2 d\eta = (2/3) \left[\eta^3 \right]_0^1 = 2/3 \\ \langle z^2 \rangle &= \int_0^1 2\eta^3 d\eta = (2/4) \left[\eta^4 \right]_0^1 = 1/2 \end{aligned}$$

$$\text{Var}(z) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}, \quad \text{S.D.} = \frac{1}{\sqrt{18}} = 0.24$$

Example Desorbing atom

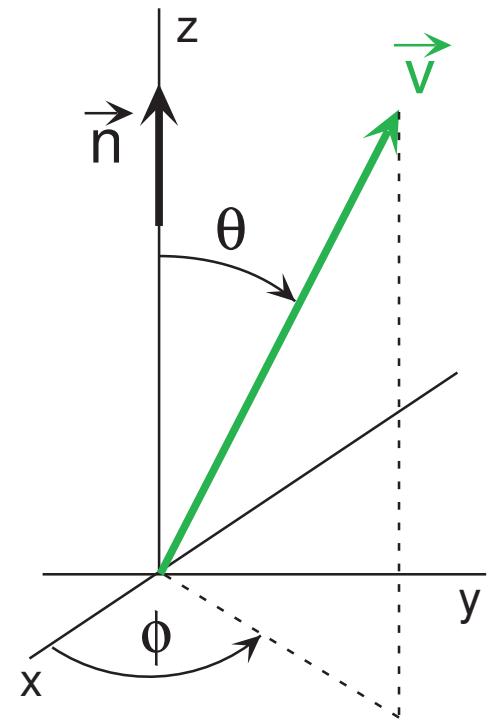
$$p(v, \theta, \phi) = p(v) p(\theta) p(\phi)$$

$$p(v) = (1/2\sigma^4) v^3 \exp[-v^2/2\sigma^2]$$

$$p(\theta) = 2 \sin \theta \cos \theta$$

$$p(\phi) = 1/2\pi$$

Find $p(v_z)$

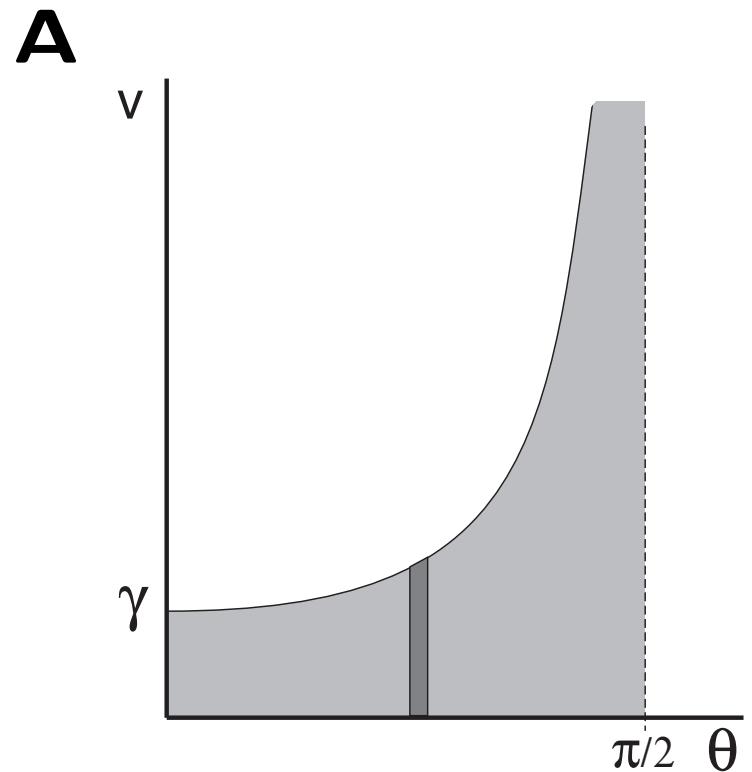


leaving the surface

$$v_z = v \cos \theta$$

$$v \cos \theta < \gamma$$

$$\Rightarrow v < \gamma / \cos \theta$$



8.044 L3B14

B

$$\begin{aligned} P_{v_z}(\gamma) &= \int_0^{\pi/2} \int_0^{\gamma/\cos\eta} p_v(\zeta) p_\theta(\eta) d\zeta d\eta \\ &= \int_0^{\pi/2} p_\theta(\eta) \left[\int_0^{\gamma/\cos\eta} p_v(\zeta) d\zeta \right] d\eta \end{aligned}$$

C

$$p_{v_z}(\gamma) = \frac{dP_{v_z}(\gamma)}{d\gamma} = \int_0^{\pi/2} p_\theta(\eta) \left[\frac{1}{\cos\eta} p_v\left(\frac{\gamma}{\cos\eta}\right) \right] d\eta$$

$$p_{V_z}(\gamma) =$$

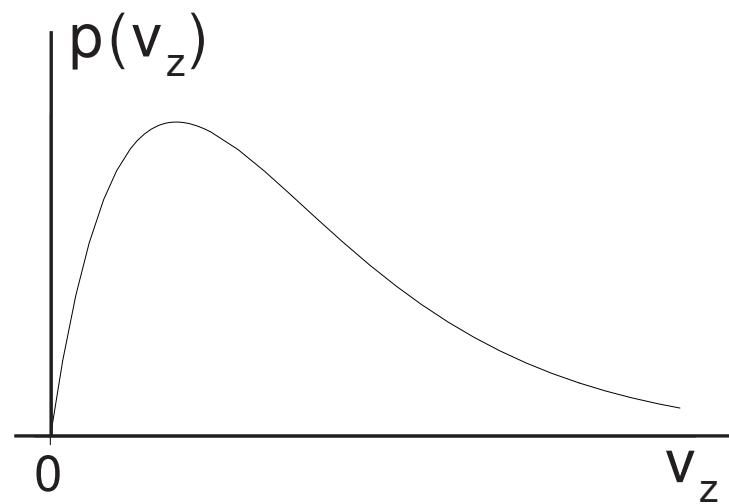
$$\int_0^{\pi/2} (2 \sin \eta \cos \eta) \left[\frac{1}{\cos \eta} \frac{1}{2\sigma^4} \left(\frac{\gamma}{\cos \eta} \right)^3 \exp \left[-\frac{1}{2\sigma^2} \frac{\gamma^2}{\cos^2 \eta} \right] d\eta \right]$$

Let $\frac{1}{2\sigma^2} \frac{\gamma^2}{\cos^2 \eta} \equiv X$

$$dX = -\frac{1}{\sigma^2} \frac{\gamma^2}{\cos^3 \eta} (-\sin \eta) d\eta$$

$$\eta = 0 \Rightarrow X = \gamma^2 / 2\sigma^2; \quad \eta = \pi/2 \Rightarrow X = \infty$$

$$\begin{aligned}
 p_{v_z}(\gamma) &= \frac{\gamma}{\sigma^2} \int_{\gamma^2/2\sigma^2}^{\infty} e^{-X} dX = -\frac{\gamma}{\sigma^2} \left[e^{-X} \right]_{\gamma^2/2\sigma^2}^{\infty} \\
 &= \frac{\gamma}{\sigma^2} \exp[-\gamma^2/2\sigma^2] \quad \gamma > 0
 \end{aligned}$$



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8.044 Statistical Physics I
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