Sums of Random Variables

Consider n RVs x_i and let $s \equiv \sum_{i=1}^{n} x_i$.

If the RVs are statistically independent, then

$$< s > \; = \; \sum_i < x_i >$$

$$\text{Var}(s) \; = \; \sum_i \text{Var}(x_i)$$

- The individual $p(x_i)$ could be quite different

- Both continuous and discrete RVs could be present

- True for any n

- Even if one RV dominates the sum

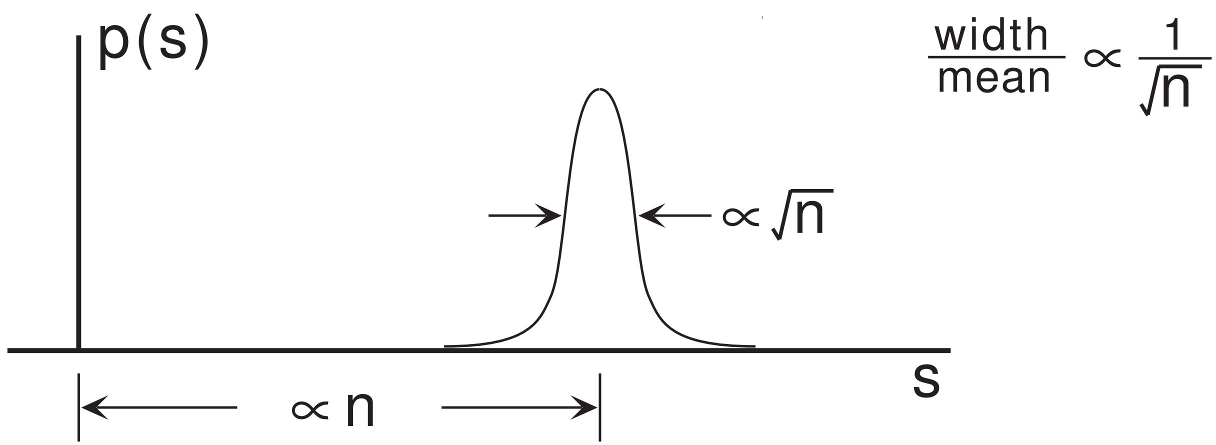
Results have a special meaning when

1) The means are finite ($\neq 0$)

2) The variances are finite ($\neq \infty$)

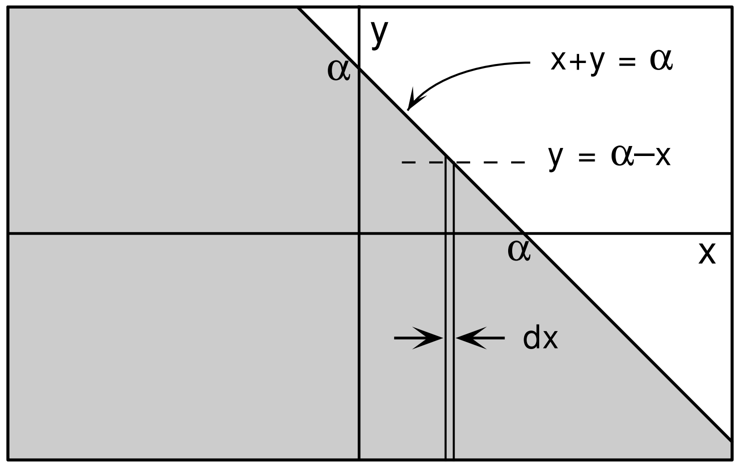
3) No subset dominates the sum

4) n is large



Given $p(x, y)$, find $p(s \equiv x + y)$

A



B

$$P_s(\alpha) = \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\alpha-\zeta} d\eta \, p_{x,y}(\zeta, \eta)$$

C $$p_s(\alpha) = \int_{-\infty}^{\infty} d\zeta \; p_{x,y}(\zeta, \alpha - \zeta)$$

This is a general result; x and y need not be S.I.

Application to the Jointly Gaussian RVs in Section 2 shows that $p(s)$ is a Gaussian with zero mean and a Variance $= 2\sigma^2(1 + \rho)$.

In the special case that x and y are S.I.

$$p_s(\alpha) = \int_{-\infty}^{\infty} d\zeta \, p_x(\zeta) \, p_y(\alpha{-}\zeta) = \int_{-\infty}^{\infty} d\zeta' \, p_x(\alpha{-}\zeta') \, p_y(\zeta')$$

The mathematical operation is called "convolution".

$$p \otimes q \equiv \int_{-\infty}^{\infty} p(z)q(x-z)dz = f(x).$$

Example

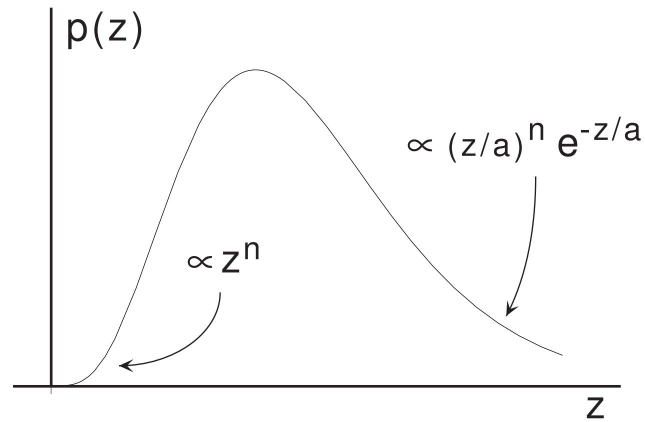
Given:

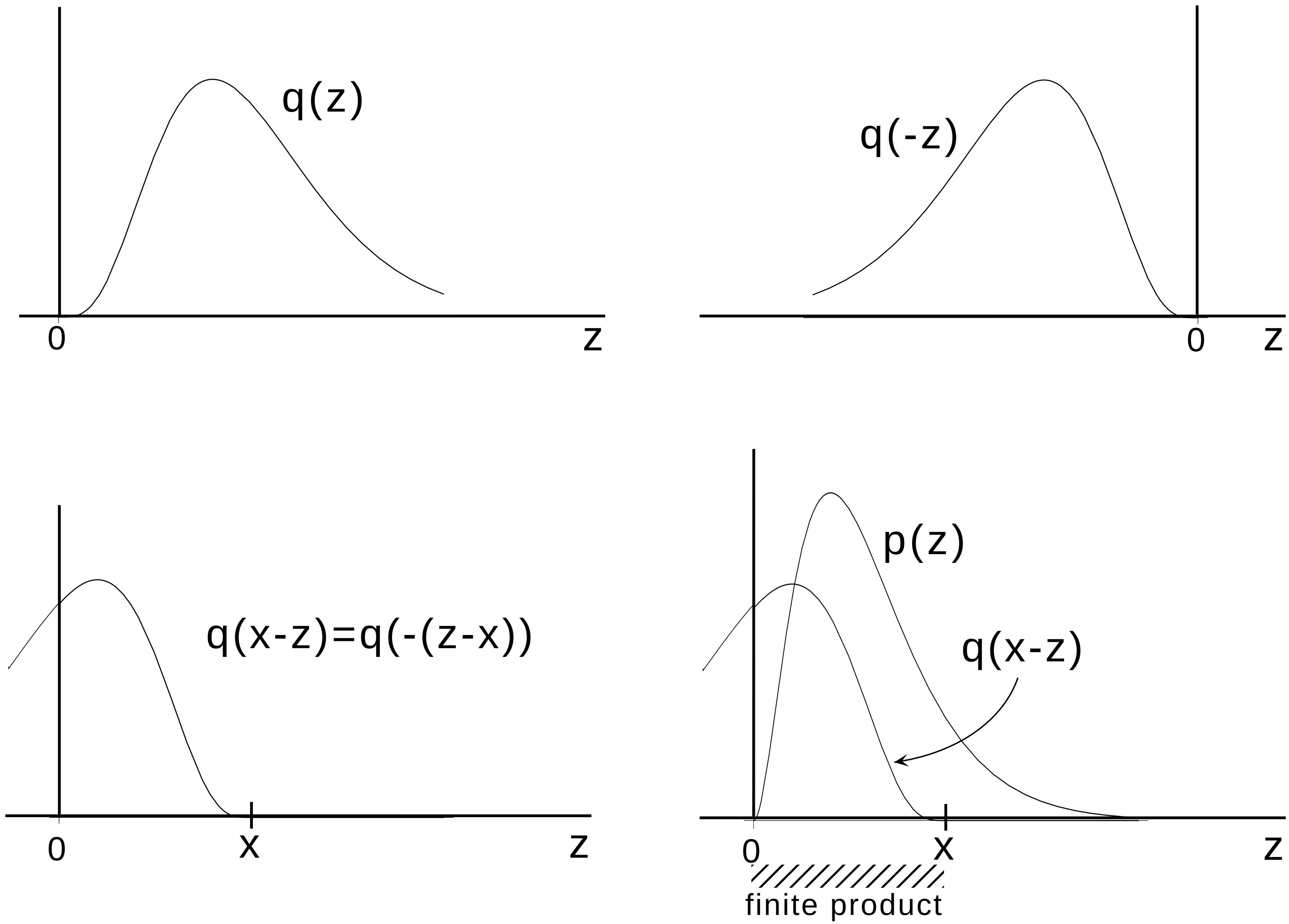
$$p(z) = \frac{1}{n!a}(z/a)^n \exp(-z/a)$$

$$q(z) = \frac{1}{m!a}(z/a)^m \exp(-z/a)$$

$$0 < z \text{ and } n, m = 0, 1, 2, \cdots$$

Find: $p \otimes q$




q(z)
0
z
q(-z)
0
z
q(x-z)=q(-(z-x))
0
x
z
p(z)
q(x-z)
0
x
z
finite product

$$p \otimes q = \frac{1}{n!m!} \frac{1}{a^2} \int_0^x \left(\frac{z}{a}\right)^n \left(\frac{x-z}{a}\right)^m e^{-z/a} \, e^{-(x-z)/a} \, dz$$

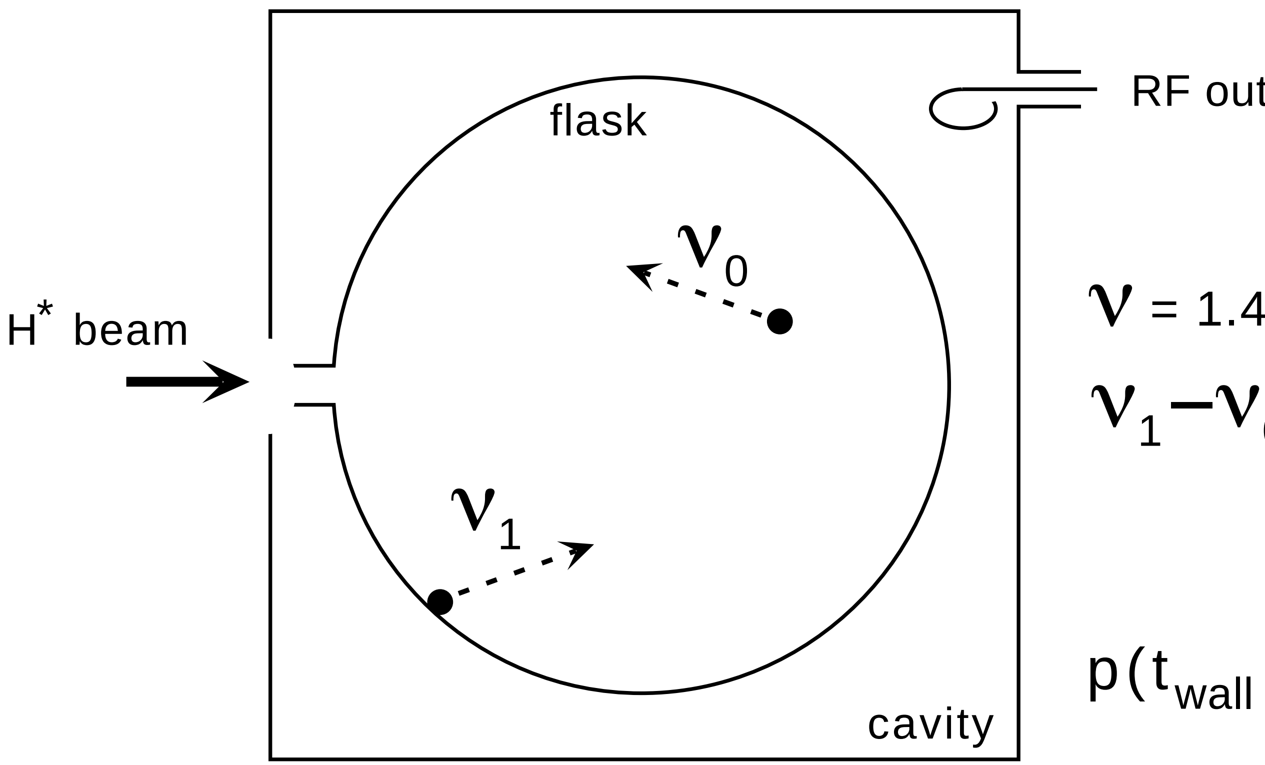
$$= \frac{1}{n!m!} \frac{1}{a} \left(\frac{1}{a}\right)^{n+m+1} e^{-x/a} \int_0^x z^n \, (x-z)^m \, dz$$

$$= \frac{1}{n!m!} \frac{1}{a} \left(\frac{x}{a}\right)^{n+m+1} e^{-x/a} \underbrace{\int_0^1 \zeta^n \, (1-\zeta)^m \, d\zeta}_{\frac{n!m!}{(n+m+1)!}}$$

$$p \otimes q = \frac{1}{(n+m+1)!} \frac{1}{a} \left(\frac{x}{a}\right)^{n+m+1} e^{-x/a}$$

a function of the same class

Example Atomic Hydrogen Maser



ν = 1.4……. GHz

$\nu_1 - \nu_0$ about 10 KHz

$p(t_{wall}\,|\,n \text{ stays}) = ?$

$$t_{\text{wall}} \text{ (given } n \text{ stays)} = \sum_{i=1}^{n} t_i$$

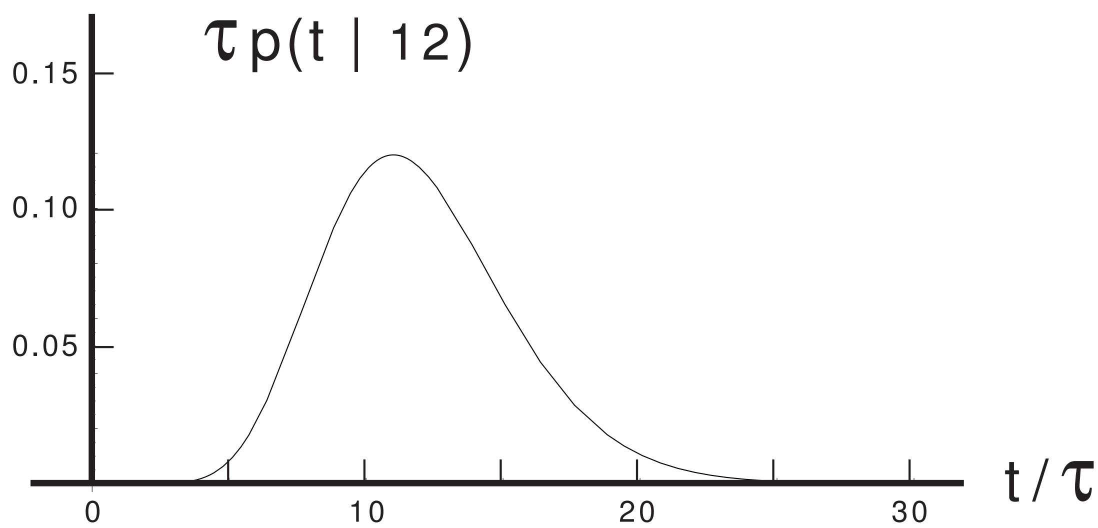
$t_i \equiv$ duration of i_{th} stay on wall. Each stay is S.I.

$$p(t\,|1) = (1/\tau)\, e^{-t/\tau}$$

$$p(t\,|2) = p(t\,|1) \otimes p(t\,|1) = (1/\tau)(t/\tau)\, e^{-t/\tau}$$

$$p(t\,|3) = p(t\,|2) \otimes p(t\,|1) = (1/2)(1/\tau)(t/\tau)^2\, e^{-t/\tau}$$

$$p(t\,|n) = \frac{1}{(n-1)!}\,\frac{1}{\tau}\,\left(\frac{t}{\tau}\right)^{n-1}\,e^{-t/\tau}$$



Facts about sums of RVs

- Exact expressions for $< s >$ and Var(s) if S.I.

- $p(s) = p(x) \otimes p(y)$ if S.I.

- $p(s)$ slightly more complicated if not S.I.

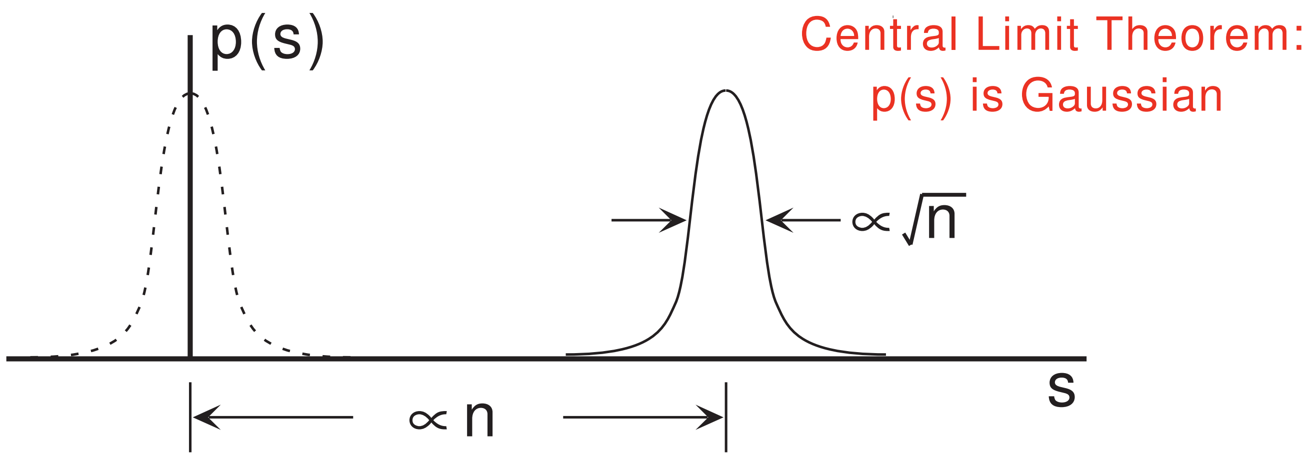
- $\otimes$ usually changes functional form

- But not always

- Fourier techniques are very useful

Very important special case: Central Limit Theorem

- RVs are S.I.
- All have identical densities $p(x_i)$
- Var(x) is finite but $< x >$ could be zero
- n is large



If x is continuous

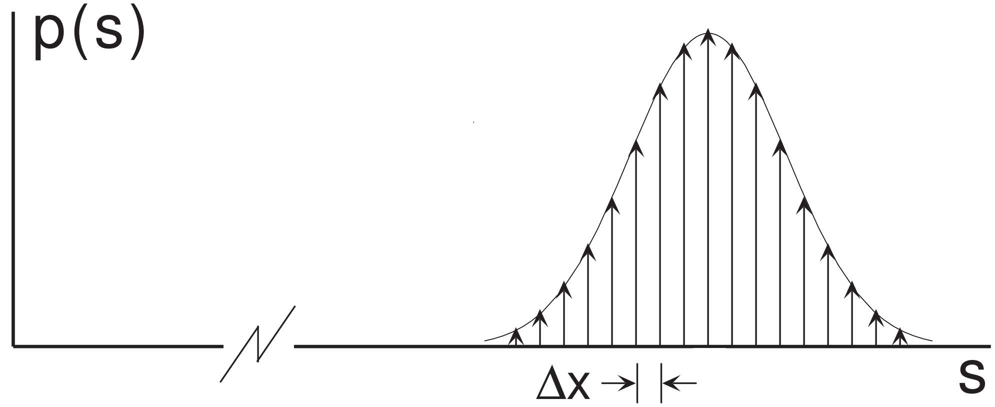
$$p(s) = \frac{1}{\sqrt{2\pi\sigma^2}}\, \mathrm{e}^{-(s-<s>)^2/2\sigma^2}$$

$$< s >= n < x >$$

$$\sigma^2 = n\, \sigma_x^2$$

If x is discrete in equal steps of Δx

$$p(s) = \sum_i \underbrace{\frac{\Delta x}{\sqrt{2\pi\sigma^2}}\, e^{-(s-<s>)^2/2\sigma^2}}_{\text{envelope}} \underbrace{\delta(s - i\,\Delta x)}_{\text{comb}}$$



Non-rigorous extensions of the Central Limit Theorem

- The Gaussian can be a good practical approximation for modest values of n.

- The Central Limit Theorem may work even if the individual members of the sum are not identically distributed.

- The requirement that the variables be statistically independent may even be waived in some cases, particularly when n is very large

8.044 Statistical Physics I
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