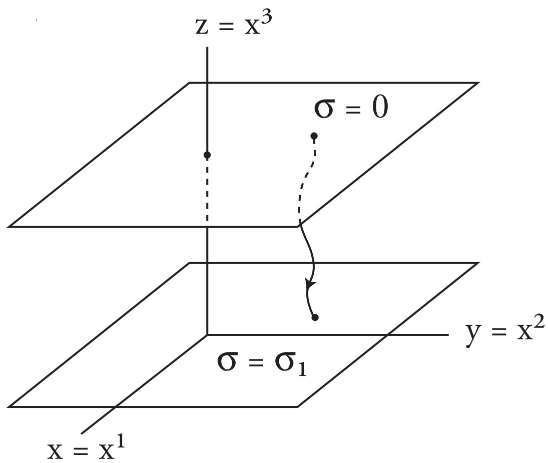
8.251 2005 Midterm Solutions

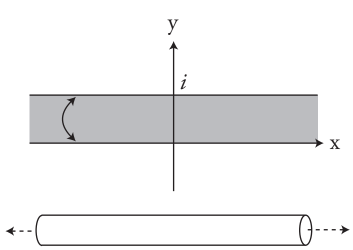
Question 1



$$\left.\begin{array}{ll} X^0(\tau,0), & X^0(\tau,\sigma_1) \\ X^1(\tau,0), & X^1(\tau,\sigma_1) \\ X^2(\tau,0), & X^2(\tau,\sigma_1) \end{array}\right\} \text{all "Free"}$$

$$\left.\begin{array}{l} X^3(\tau,0) = a \\ X^3(\tau,\sigma_1) = 0 \end{array}\right\} \text{Dirichlet}$$

Question 2



(a)

$$z \sim z + i$$

$\mathcal{F}: \quad 0 \leq y < 1 \qquad \mathcal{M}: \quad$ infinite cylinder

(b)

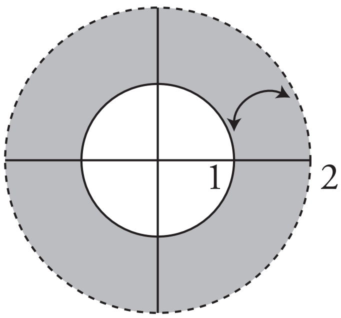
$$z \sim 2z$$

This scales z, keeping the argument unchanged.

$$\mathcal{F} : 1 \leq |z| < 2$$

$\mathcal{M}$: <u>torus</u>

Technically we should also include the point $z = 0$.



Question 3

$$S = \int d^3x \, (A_0 F_{12} + A_1 F_{20} + A_2 F_{01})$$

$$\delta S = \int d^3x \, (\delta A_0 F_{12} + A_1(\partial_2 \delta A_0) - A_2 \partial_1(\delta A_0))$$

$$= \int d^3x \, (\delta A_0 F_{12} - \delta A_0 \partial_2 A_1 + (\partial_1 A_2)\delta A_0)$$

$$\delta S = \int d^3x \delta A_0 (2F_{12})$$

EOM: $\boxed{F_{12} = 0}$

Question 4

$$\delta = 8\pi G T_0$$

δ should have no units with suitable c's and $\hbar$'s

(a) Units of GT_0?

$$F = \frac{GM^2}{r^2}$$

$$[G] = [F]\frac{L^2}{M^2} = M\frac{L}{T^2}\frac{L^2}{M^2}$$

$$[G] = \frac{L^3}{T^2 M}$$

$$[T_0] = \frac{ML}{T^2}$$

$$\rightarrow [GT_0] = \frac{L^4}{T^4} \quad \text{thus need a } c^4$$

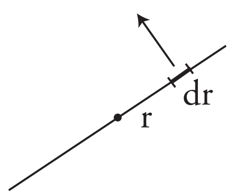
$$\boxed{\delta = \frac{8\pi G T_0}{c^4}}$$

(b) Mass density $\mu_0 = \frac{T_0}{c^2}$

$$\frac{T_0}{c^2} = \delta \cdot \frac{c^2}{8\pi G} = \left(0.5 \frac{\cancel{2\pi}}{360 \cdot 60 \cdot 60}\right) \frac{\left(3 \times 10^8\right)^2}{\cancel{2\pi} \cdot 4 \cdot (6.67 \times 10^{-11})}$$

$$\frac{T_0}{c^2} = 1.30 \times 10^{20} \frac{\text{kg}}{\text{m}}!$$

Question 5



(a)

$$v = \omega r$$

$$dp = \frac{(dm)\omega r}{\sqrt{1 - \frac{\omega^2 r^2}{c^2}}}$$

$$\text{but} \quad dm = \frac{dE}{c^2} = \frac{T_0 dr}{c^2}$$

$$\boxed{dp = \frac{T_0 \omega}{c^2} \frac{r dr}{\sqrt{1 - \frac{\omega^2 r^2}{c^2}}}}$$

$$\boxed{dJ = r dp = \frac{T_0 \omega}{c^2} \frac{r^2 dr}{\sqrt{1 - \frac{\omega^2 r^2}{c^2}}}}$$

(b)

$$J = 2 \int_0^{l/2} \frac{T_0 \omega}{c^2} \frac{r^2 dr}{\sqrt{1 - \frac{\omega^2 r^2}{c^2}}}$$

$$\text{let:} \quad \frac{\omega r}{c} = x \quad r = \frac{c}{\omega} x$$

$$J = 2 \frac{T_0 \omega}{c^2} \cdot \int_0^1 \left(\frac{c}{\omega}\right)^3 \frac{x^2 dx}{\sqrt{1 - x^2}}$$

$$J = 2\frac{T_0 c}{\omega^2}\underbrace{\int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}}}_{\frac{\pi}{4}} = \frac{\pi}{2}\frac{T_0 c}{\omega^2}$$

$$\omega\frac{l}{2} = c \quad \omega = \frac{2c}{l}$$

$$\omega = \frac{2c}{\frac{2}{\pi}\frac{E}{T_0}} = \frac{\pi T_0 c}{E}$$

$$J = \frac{\pi}{2}\frac{T_0 c E^2}{\pi^2 T_0^2 c^2} = \frac{E^2}{2\pi T_0 c}$$

Question 6

$$\vec{X}(t,\sigma) = \frac{1}{2}\left(\vec{F}(u) + \vec{G}(\vec{v})\right)$$

(a)

$$\vec{X}(t,\sigma+\sigma_1) = \vec{X}(t,\sigma)$$

$$\vec{F}(u,+\sigma_1) + \vec{G}(v-\sigma_1) = \vec{F}(u) + \vec{G}(v)$$

$$\boxed{\vec{F}(u+\sigma_1) - \vec{F}(u) = \vec{G}(v) - \vec{G}(v-\sigma_1)}$$

(b)

$$\vec{f}(u+\sigma_1) + \vec{\alpha}(u+\sigma_1) - \vec{f}(u) - \vec{\alpha}(u) = \vec{g}(v) + \vec{\beta}v - \left(\vec{g}(v-\sigma_1) + \vec{\beta}(v-\sigma_1)\right)$$

$$\vec{\alpha}\sigma_1 = \beta\vec{\sigma_1} \quad \rightarrow \quad \boxed{\vec{\alpha} = \vec{\beta}}$$

$$\vec{X}(t,\sigma) = \frac{1}{2}\left(\vec{f}(u) + \vec{\alpha}u + \vec{g}(u) + \vec{\alpha}v\right)$$

$$= \frac{1}{2}\vec{\alpha}(u+v) + \frac{1}{2}\left(\vec{f}(u) + \vec{g}(v)\right)$$

$$\boxed{\vec{X}(t,\sigma) = \vec{\alpha}ct + \frac{1}{2}\left(\vec{f}(u) + \vec{g}(v)\right)}$$

(c)

$$\vec{\mathcal{P}^\tau} = \frac{T_0}{c^2}\frac{\partial\vec{x}}{\partial t}$$

$$\vec{\mathcal{P}^\tau} = \frac{T_0}{c}\vec{\alpha} + \frac{T_0}{2c}\left(\vec{f'}(ct+\sigma) + \vec{g'}(ct-\sigma)\right)$$

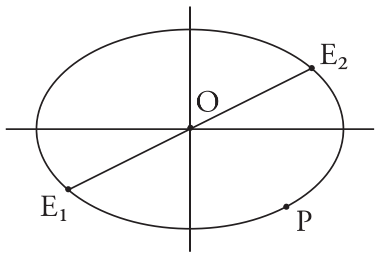
$$\vec{P}\int_0^{\sigma_1}\vec{\mathcal{P}^\tau}d\sigma = \frac{T_0}{c}\sigma_1\vec{\alpha} + \underbrace{\frac{T_0}{2c}\int_0^{\sigma_1}\frac{\partial}{\partial\sigma}\left(\vec{f}(ct+\sigma) - \vec{g}(ct-\sigma)\right)d\sigma}_{0,\text{ by periodicity of } f \text{ and } g}$$

$$\boxed{\vec{P} = \frac{T_0\sigma_1}{c}\vec{\alpha}}$$

Since $\sigma_1 = \frac{E}{T_0}$, we can also write:

$$\boxed{\vec{P} = \frac{E}{c}\vec{\alpha}}$$

Question 7



Consider an arbitrary point P on the string. If each crawler travels $L/4$, the two endpoints E_1 and E_2 are separated along the ellipse by $L/2$. Moreover, they are opposite, and the midpoint of $\overline{E_1E_2}$ is the origin. Thus P will go to the origin at $t = \frac{L}{4}\frac{1}{c}$. Since P is arbitrary, the ellipse collapses to the origin after time $\left(\frac{L}{4}\right)\frac{1}{c}$.

Comment: You may convince yourself that any curve $\mathcal{C}$ with inversion symmetry in the plane (if $\vec{a} \in \mathcal{C}$, then $(-\vec{a}) \in \mathcal{C}$) will collapse to zero size.