8.251 Test

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Only personal 2-page notes allowed. **Formula sheet on the last page.** Test duration: 2 hours.

PROBLEM 1. (15 points) Conserved current for a complex scalar field.
Consider the following action for a **complex** scalar field ϕ

$$S = \int d^D x \mathcal{L}\,, \qquad \mathcal{L} = -\frac{1}{2}\,\eta^{\alpha\beta}\,(\partial_\alpha \phi^*)\,(\partial_\beta \phi)\,,$$

where ϕ^* denotes the complex conjugate of ϕ.

(a) Assume we change ϕ as

$$\phi(x) \to e^{i\theta}\phi(x),$$

with θ an arbitrary real constant. What is the corresponding change in ϕ^*? Verify that these changes are a symmetry of the theory.

(b) Write the corresponding infinitesimal transformations $\delta\phi = \ldots$ and $\delta\phi^* = \ldots$ that arise when $\theta = \epsilon$ is small. Use these expressions to construct the conserved current j^α associated with the symmetry in (a). In using the formula for the conserved current it is possible to treat ϕ and ϕ^* as independent fields.

(c) Write $\phi(x) = A(x) + iB(x)$, where A and B are now real scalar fields. What is j^α in terms of A and B?

Useful formula: For a field transformation $\delta\phi^a = \epsilon \ldots$ the current is $\epsilon j^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\delta\phi^a$.

PROBLEM 2. (25 points) Classical closed string motion in the light-cone gauge.

A closed string described in light-cone gauge is performing motion in the (x^2, x^3) plane:

$$X^1(\tau,\sigma) = 0\,, \qquad X^2(\tau,\sigma) = a\cos\sigma\,\cos\tau\,, \qquad X^3(\tau,\sigma) = a\sin\sigma\,\cos\tau\,,$$

where a, assumed known, is a positive constant with units of length (recall $\sigma \in [0, 2\pi]$).

(a) Calculate $X^-(\tau,\sigma)$ using the information about the transverse motion and $x_0^- = 0$.

(b) Use $X^1(\tau,\sigma) = 0$ to solve for p^+ in terms of a and α'.

(c) Calculate the mass-squared M^2 of this string.

Useful formulas: For closed strings in light-cone gauge:

$$X^+ = \alpha' p^+ \tau\ , \quad \text{and} \quad \dot{X}^- \pm X^{-\prime} = \frac{1}{2\alpha' p^+}\,(\dot{X}^I \pm X^{I\prime})^2\ .$$

PROBLEM 3. Three short questions (20 points).

(a) Consider the number of physical degrees of freedom needed to describe a graviton in a D-dimensional spacetime. Compare this number to the number of physical degrees of freedom needed to describe a graviton together with a Maxwell field and a scalar in $D-1$ dimensions.

(b) Calculate the following commutators in the light-cone gauge quantization of the particle:

$$[x^+, p^-] = \dots\,, \qquad [p^-, x^I] = \dots\,, \qquad [x_0^-, p^+] = \dots$$

(c) Express the open string state

$$L_{-1}^{\perp} L_{-1}^{\perp} |p^+, \vec{p}_T\rangle$$

in terms of normal-ordered oscillators acting on the ground state.

PROBLEM 4. (40 points) Dp-branes and orientifolds.

In order to discuss a Dp-brane we split the string coordinates into

$$X^+, X^-, \{X^i\},\ \{X^a\}\,, \quad i = 2, \dots, p, \quad a = p+1, \dots, d$$

Note that we still use the light-cone gauge, but this time the transverse coordinates X^I split into two groups, the coordinates X^i along the brane and the coordinates X^a orthogonal to the brane. X^1 is taken as one of the coordinates along the brane.

The mode expansions for the coordinates X^i work just the same way as those for the X^I coordinates (both satisfy Neumann boundary conditions at the endpoints). The oscillators associated with X^i are called α_n^i. As in the previous case of completely free boundary conditions, the constraints $(\dot{X} \pm X')^2 = 0$ and the light-cone gauge condition $X^+ = 2\alpha' p^+ \tau$ can be used to solve for the derivatives of X^-:

$$\frac{1}{p^+} \sum_{n \in \mathbb{Z}} L_n^{\perp} e^{-in(\tau \pm \sigma)} \equiv \dot{X}^- \pm X^{-\prime} = \frac{1}{4\alpha' p^+} \left\{ (\dot{X}^i \pm X^{i\prime})^2 + (\dot{X}^a \pm X^{a\prime})^2 \right\}. \tag{1}$$

The derivatives of the X^i coordinates are expanded in the same way as we previously expanded the X^I coordinates, so

$$\dot{X}^i \pm X^{i\prime} = \sqrt{2\alpha'} \sum_{n \in \mathbb{Z}} \alpha_n^i e^{-in(\tau \pm \sigma)} \equiv A^i(\tau, \pm\sigma) \qquad \text{for } \sigma \in [0, \pi]\,. \tag{2}$$

The canonical commutation relations for X^i and $\mathcal{P}^{\tau j} = \dot{X}^i/(2\pi\alpha')$ then imply that

$$\left[A^i(\tau, \sigma), A^j(\tau, \sigma')\right] = 4\pi i \alpha'\, \delta^{ij}\, \frac{d}{d\sigma} \delta(\sigma - \sigma') \qquad \text{for } \sigma \in [-\pi, \pi]\,. \tag{3}$$

These relations in turn imply that the α_n^i obey the usual commutation relations, $\left[\alpha_n^i, \alpha_m^j\right] = n\, \delta_{m+n,0}\, \delta^{ij}$.

The X^a coordinates, however, have to be treated differently. Assuming the brane is located at $x^a = 0$, the mode expansion for the coordinates $X^a(\tau, \sigma)$ must satisfy the Dirichlet boundary condition $X^a(\tau, \sigma_*) = 0$ for $\sigma_* = 0, \pi$. We take

$$X^a(\tau, \sigma) = \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n}\, \alpha_n^a e^{-in\tau} \sin n\sigma\,. \tag{4}$$

Note that there is no momentum p^a nor a conjugate coordinate x^a in the above expansion.

(a) Calculate $X^{a\prime} \pm \dot{X}^a$ and comment on the relation to (2). State the equal-time commutation relations for the X^a and the $\mathcal{P}^{\tau b}$. In analogy to (2), define

$$X^{a\prime}(\tau,\sigma) \pm \dot{X}^a(\tau,\sigma) \equiv A^a(\tau, \pm\sigma)\,.$$

Write the commutation relations that $A^a(\tau,\sigma)$ must obey. The correct commutation relations imply that the α_n^a obey the usual commutation relations, $\left[\alpha_n^a, \alpha_m^b\right] = n\,\delta_{m+n,0}\,\delta^{ab}$.

(b) Use Eq. (1) to calculate the new operator $L_0^\perp$, defined to be normal-ordered. The answer should involve both the α_n^i and α_n^a oscillators. As before we have $2\alpha' p^+ p^- = L_0^\perp - 1$. Write the new mass-squared formula.

(c) What are the labels of the ground states of the theory? List all the states with $N^\perp = 0, 1,$ and 2. From the viewpoint of an observer who lives on the brane, what fields correspond to the massless states of the theory?

An orientifold plane is obtained when one truncates the theory to the states invariant under an operation Ω_p that reverses the orientation of all strings and reflects the coordinates normal to the plane. (A reflection of a coordinate x is a map $x \to -x$.) Consider adding an orientifold Op-plane coincident with our Dp-brane.

(d) Fill in the blanks in the right-hand sides for the action of Ω_p on the string coordinates

$$\Omega_p\, X^i(\tau,\sigma)\,\Omega_p^{-1} = \ldots \qquad \Omega_p\, X^a(\tau,\sigma)\,\Omega_p^{-1} = \ldots$$

Give the action of Ω_p on the oscillators α_n^i and α_n^a.

(e) What are the states of the orientifold theory for $N^\perp = 0,\ 1$ and 2? Write the operator Ω_p in terms of the operator $N^\perp$.

Possibly Useful Formulas

Light-Cone Coordinates: $x^{\pm} = \frac{1}{\sqrt{2}}\left(x^0 \pm x^1\right)$.

Relativistic Point Particle in Light-Cone Coordinates: $x^+ = \frac{p^+}{m^2}\,\tau\,,\quad p^- = \frac{1}{2p^+}\left(p^I p^I + m^2\right)$.

Slope Parameter: $\alpha' = \frac{1}{2\pi T_0}$.

Light-Cone Gauge:

$$X^+ = \beta\alpha' p^+\,\tau, \quad \text{where} \quad \beta = \begin{cases} 2 & \text{for open strings} \\ 1 & \text{for closed strings}\,, \end{cases}$$

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'}\,\dot{X}^\mu\,, \qquad \mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}\,X^{\mu\prime}\,,$$

$$(\dot{X} \pm X')^2 = 0 \quad \Longrightarrow \quad \dot{X}^- \pm X^{-\prime} = \frac{1}{2\beta\alpha' p^+}\left(\dot{X}^I \pm X^{I\prime}\right)^2\,,$$

$$\ddot{X}^\mu - X^{\mu\prime\prime} = 0\,.$$

Open String Expansion: $X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\,e^{-in\tau}\,\cos n\sigma$.

Closed String Expansion: $X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(\alpha_n^\mu\,e^{in\sigma} + \bar{\alpha}_n^\mu\,e^{-in\sigma}\right)$.

Momentum: $\alpha_0^\mu = \sqrt{2\alpha'}\,p^\mu$ (open strings) , $\qquad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\,p^\mu$ (closed strings).

Commutators, Creation and Annihilation Operators:

$$\left[\alpha_n^I\,,\,\alpha_m^J\right] = n\delta_{m+n,0}\,\delta^{IJ}\,, \qquad \left[x_0^I\,,\,\alpha_0^J\right] = \sqrt{2\alpha'}\,i\,\delta^{IJ}\,, \qquad \left[x_0^I\,,\,\alpha_n^J\right] = 0 \text{ if } n \neq 0\,,$$

$$\text{for } n \geq 1: \quad \alpha_n^I = \sqrt{n}\,a_n^I\,, \qquad \alpha_{-n}^I = \sqrt{n}\,a_n^{I\dagger}\,, \qquad \left[a_m^I\,,\,a_n^{J\dagger}\right] = \delta^{IJ}\,\delta_{mn}\,.$$

Virasoro Algebra (Open Strings):

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}\,p^+}L_n^\perp\,, \qquad \text{where} \quad L_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty}\alpha_{n-p}^I\,\alpha_p^I\,, \quad \left[L_m^\perp\,,\,\alpha_n^J\right] = -n\alpha_{m+n}^J\,,$$

$$\left[L_m^\perp\,,\,L_n^\perp\right] = (m-n)L_{m+n}^\perp + \frac{1}{12}m(m^2-1)(D-2)\delta_{m+n,0}\,, \quad \left[L_m^\perp\,,\,x_0^I\right] = -i\sqrt{2\alpha'}\,\alpha_m^I\,,$$

$$L_0^\perp = \alpha'\,p^I\,p^I + N^\perp\,, \qquad N^\perp = \sum_{p=1}^{\infty}\alpha_{-p}^I\,\alpha_p^I = \sum_{n=1}^{\infty}n\,a_n^{I\dagger}\,a_n^I\,,$$

$$M^2 = -p^2 = 2\,p^+\,p^- - p^I\,p^I = \frac{1}{\alpha'}\left(N^\perp - 1\right)\,.$$

Virasoro Algebra (Closed Strings):

$$\alpha_n^- = \frac{1}{p^+}\sqrt{\frac{2}{\alpha'}}\,L_n^\perp\,, \quad \bar{\alpha}_n^- = \frac{1}{p^+}\sqrt{\frac{2}{\alpha'}}\,\bar{L}_n^\perp\,, \quad L_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty}\alpha_{n-p}^I\,\alpha_p^I\,, \quad \bar{L}_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty}\bar{\alpha}_{n-p}^I\,\bar{\alpha}_p^I\,,$$

$$L_0^\perp = \frac{\alpha'}{4}\,p^I\,p^I + N^\perp\,, \quad N^\perp = \sum_{p=1}^{\infty}\alpha_{-p}^I\,\alpha_p^I = \sum_{n=1}^{\infty}n\,a_n^{I\dagger}\,a_n^I\,,$$

$$\bar{L}_0^\perp = \frac{\alpha'}{4}\,p^I\,p^I + \bar{N}^\perp\,, \quad \bar{N}^\perp = \sum_{p=1}^{\infty}\bar{\alpha}_{-p}^I\,\bar{\alpha}_p^I = \sum_{n=1}^{\infty}n\,\bar{a}_n^{I\dagger}\,\bar{a}_n^I\,,$$

$$M^2 = -p^2 = 2\,p^+\,p^- - p^I\,p^I = \frac{2}{\alpha'}\left(N^\perp + \bar{N}^\perp - 2\right)\,, \quad \bar{N}^\perp = N^\perp\,.$$