Lecture 11 - Topics

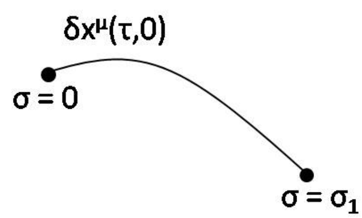
- Static gauge, transverse velocity and string action
- Motion of free open string endpoints

Reading: Section 6.6 - 6.9

$$\frac{\partial \mathcal{P}^{\tau\mu}}{\partial \tau} + \frac{\mathcal{P}^{\sigma\mu}}{\partial \sigma} = 0$$

$$\delta S = \int d\tau [\delta x^\mu \mathcal{P}^\sigma_\mu]_0^{\sigma_1} - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left(\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\tau_\mu}{\partial \sigma} \right) \delta x^\mu$$

$$\int_{\tau_i}^{\tau_f} d\tau [\delta X^0(\tau,\sigma_1)\mathcal{P}^\sigma_0(\tau,\sigma_1) - \delta X^0(\tau,0)\mathcal{P}^\sigma_0(\tau,\sigma_1) + \delta X^1(\tau,\sigma_1)\mathcal{P}_1^{\sigma)(\tau,\sigma_1)-\delta X^1(\tau,0)\mathcal{P}^\sigma_1}$$



$$\mathcal{P}^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial x'^\mu}$$

$\sigma = 0$ and $\sigma = \sigma_1$ are the endpoints of the string. $[\tau_i, \tau_f]$ is the time interval that the string is evolving over. String described at end by $\delta x^\mu(\tau_f, \sigma)$ (σ varies from 0 to σ_1).

How do we make this variation δS vanish?

Let $\sigma_* \in \{\sigma = 0, \sigma_1\}$ (so whatever we say about σ_* applies to both $\sigma = 0$ and σ_1).

$$\delta x^\mu(\tau, \sigma_*)$$

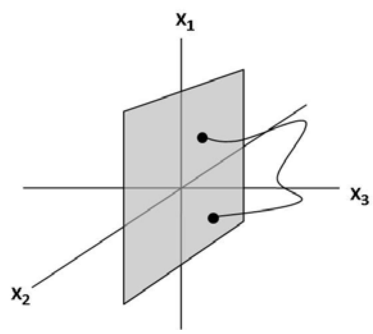
Impose a Dirichlet BC. Some $x^\mu(\tau, \sigma_*)$ is a constant oas a function of τ.

$$\frac{\partial x^\mu}{\partial \tau}(\tau, \sigma_*) = 0$$

Then $\delta x^\mu(\tau, \sigma_*) = 0$. But can't impose Dirichelt BC on $x^0(\tau, \sigma_*)$. Time always flows. Never a constant.

Impose Free BCs:
$\delta x^\mu(\tau, \sigma_*)$ arbitrary $\Rightarrow \mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0$. Must include $\mathcal{P}^\sigma_{\mu=0}(\tau, \sigma_*) = 0$ (again since time flows)

D-Branes

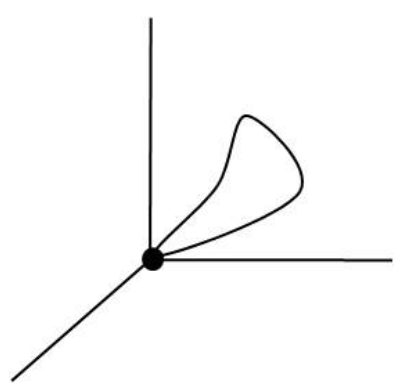


D2-Brane: 2 = number of spatial dimensions of the object
DP-Brane: P = number of spatial dimensions (with free BCs) where endpoints can move freely

BCs for motion of an open string on a D2-brane:
$x^3(\tau, \sigma_*) = 0$
$\mathcal{P}^\sigma_0(\tau, \sigma_*) = 0$
$\mathcal{P}^\sigma_1(\tau, \sigma_*) = 0$
$\mathcal{P}^\sigma_2(\tau, \sigma_*) = 0$

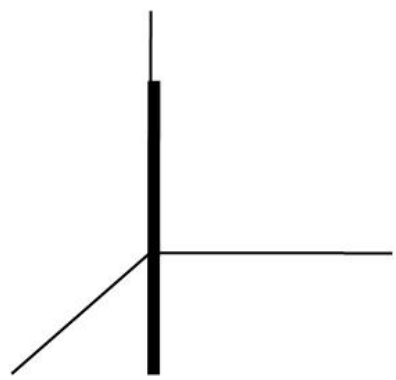
$\sigma_* = 0$ or σ_1

D0-Brane:



All strings forced to start and end at the point (string looks like a closed string but has different equations of motion because it can't move away freely)

D1-Brane:

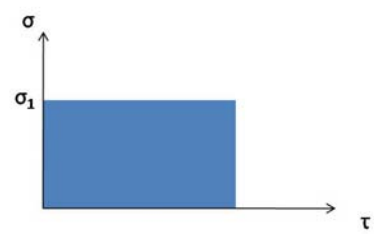


Looks like a string, but not actually one.

Can have up to Dd-Branes.

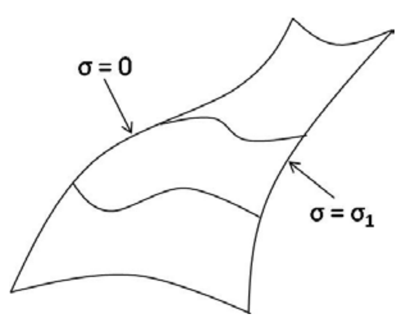
Have cartesian coordinates for:

1.

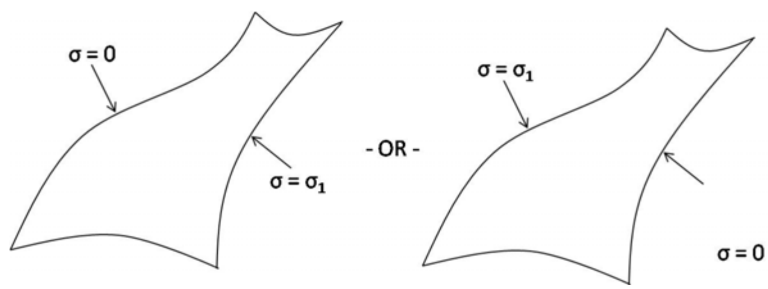


String swejpg out spacetime surface by moving through time.

2.



Actually matters whether

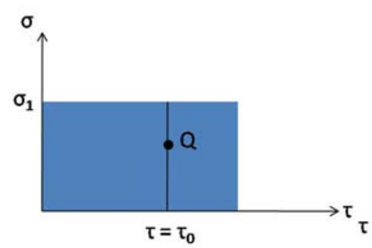


Orientation matters. Will see this later.

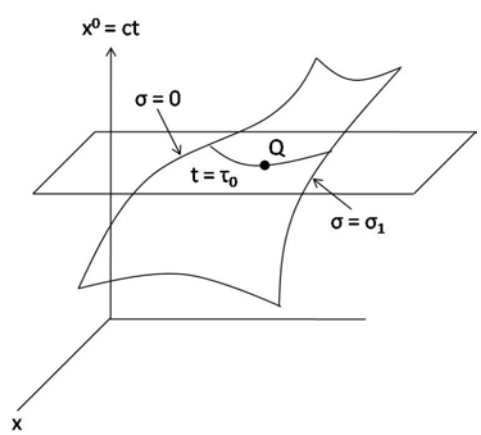
Static Gauge

Will enable us to draw lines on the surface (2) from lines on (1)

Consider line $\tau = \tau_0$ or 1



Draw on worldsheet



$$\tau(Q) = t(Q)$$
$$t = \tau$$

$$x^0(\tau,\sigma) = ct = c\tau$$

Description of Coordinates:

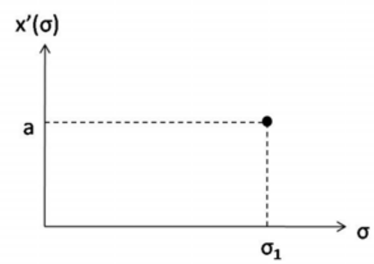
$$x^\mu(\tau,\sigma) = \{c\tau, \vec{x}(t,\sigma)\} = \{c\tau, \vec{x}(\tau,\sigma)\}$$

Remember σ is not the length of the string - it's a parameter, so σ_1 is constant. BUT the string can elongate or shorten, of course.

Some useful quantities:

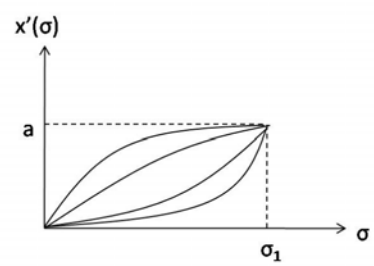
$$\dot{x}^\mu(\tau,\sigma) = (c, \partial\vec{x}/\partial t)$$
$$x^{\mu\prime}(\tau,\sigma) = (0, \partial\vec{x}/\partial\sigma)$$
$$\dot{x}\cdot x' = \partial\vec{x}/\partial t \cdot \partial\vec{x}/\partial\sigma$$
$$\dot{x}^2 = -c^2 + (\partial\vec{x}/\partial\ t)^2$$
$$x'^2 = (\partial\vec{x}/\partial\sigma)^2$$

Consider static string stretched between $x^1 = 0$ and $x^1 = a$

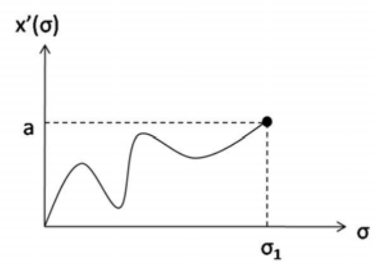


$$x^1(\tau,\sigma) = x^1(\sigma) \qquad \text{(independent of } \tau)$$

Plot $x^1(\sigma)$ vs σ



Since Nambu-Gotta action, reparam-invar, doesn't matter what path chosen as long as not e.g



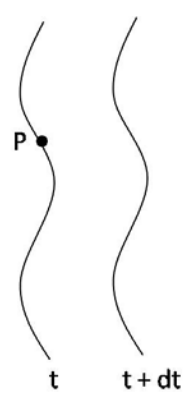
So:

$$\dot{x}^\mu = (c, 0, \vec{0}) \Rightarrow \dot{x}^2 = -c^2$$
$$x^{\mu\prime} = (0, dx^1/d\sigma, \vec{0}) \Rightarrow x'^2 = (dx^1/d\sigma)^2$$

Nambu-Gotta action:

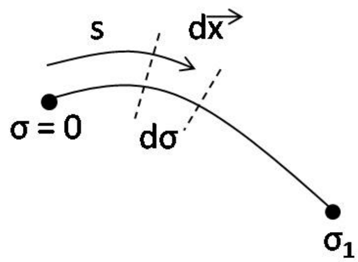
$$\begin{aligned} S &= -\frac{T_0}{c}\int d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{x}\cdot x^1)^2 - (-c^2)(\frac{dx^1}{d\sigma})^2} \\ &= -T_0 \int d\tau \int_0^{\sigma_1} d\sigma (dx^1/d\sigma) \\ &= -T_0 \int d\tau (x^1(\sigma_1) - x^1(0)) \\ &= \int_{t_i}^{t_f} dt(-T_0 a) \end{aligned}$$

Recall $S = \int (K - V)dt$. String not moving so $K = 0$. For stretched string, V = tension·distance. So physically, the tension of the string in constant. $T_0 a$: potential energy of static string stretched to length a.



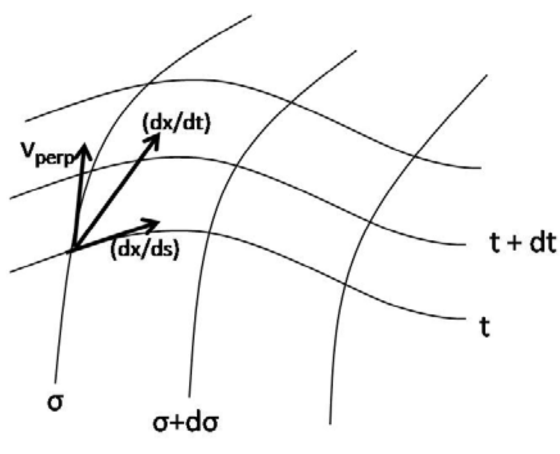
Where did point P go from t to $t + dt$?
No physical answer! So hard to talk about velocites.

But let's construct a velocity we can all agree on. Construct plane perpendicular to P. Say P'' moves to $(P')''$ where P' is the intersection of the plane with the string at $t + dt$. This is called the perpendicular velocity. (String doesn't actually neccesarily move perpendicularly, but this well-defined quantity pretends it does).



$$ds = |d\vec{x}| \to d\vec{x}/ds = 1$$

$d\vec{x}/ds$ is a unit vector tangent to the string.



$$\vec{v_\perp} = \frac{\partial \vec{x}}{\partial t} - \left(\frac{\partial \vec{x}}{\partial t} \cdot \frac{\partial \vec{x}}{\partial x}\right) \frac{\partial \vec{x}}{\partial s}$$

$$v_\perp^2 = (\partial\vec{x}/\partial t)^2 - (\partial\vec{x}/\partial t \cdot \partial\vec{x}/\partial s)^2$$

Let's simplify Nambu-Gotta action:

$$
\begin{aligned}
(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2 &= \left(\frac{\partial \vec{x}}{\partial t}\cdot\frac{\partial \vec{x}}{\partial \sigma}\right)^2 - \left(-c^2 + \left(\frac{\partial \vec{x}}{\partial t}\right)^2\right)\left(\frac{\partial \vec{x}}{\partial \sigma}\right)^2 \\
&= \left(\frac{ds}{d\sigma}\right)^2\left[\left(\frac{\partial \vec{x}}{\partial t}\cdot\frac{\partial \vec{x}}{\partial s}\right)^2 + c^2 - \left(\frac{\partial \vec{x}}{\partial t}\right)^2\right] \\
&= \left(\frac{ds}{d\sigma}\right)[c^2 - v_\perp^2]
\end{aligned}
$$

Nambu-Gotta action knew nothing mattered except perpendicular velocity. No way to tell how a point moves.

So $\sqrt{\ldots} = c\frac{ds}{d\sigma}\sqrt{1-\frac{v^2}{c^2}}$

$$
\boxed{S = -T_0\int dt\int_0^{\sigma_1} d\sigma\frac{ds}{d\sigma}\sqrt{1-\frac{v_\perp^2}{c^2}} = -T_0\int dt\int ds\sqrt{1-v_\perp^2/c^2}}
$$

Recall: $L = -m_0 c^2\sqrt{1-v^2/c^2}$.

$T_0 ds$: rest energy of small section of string.

Consider a totally free open string (no D-branes)

$$
\mathcal{P}^{\sigma\mu}(\tau,\sigma_*) = 0
$$

$$
\begin{aligned}
\mathcal{P}^{\sigma\mu} &= -\frac{T_0}{c}\frac{(\dot{x}\cdot x')\dot{x}^\mu - \dot{x}^2 x'^\mu}{\sqrt{\ldots}} \\
&= -\frac{T_0}{c^2}\frac{\left(\frac{\partial \vec{x}}{\partial t}\cdot\frac{\partial \vec{x}}{\partial s}\right) + \left(c^2 - \left(\frac{\partial x}{\partial t}\right)^2\right)\frac{\partial x^\mu}{\partial s}}{\sqrt{1-v_\perp^2/c^2}}
\end{aligned}
$$

Note magic with the $ds/d\sigma$
$\mu = 0$:

$$
\mathcal{P}^{\sigma\mu} = -\frac{T_0}{c}\frac{\frac{\partial \vec{x}}{\partial s}\cdot\frac{\partial \vec{x}}{\partial t}}{\sqrt{1-v_\perp^2/c^2}}\Big|_{\text{endpoint}} = 0
$$

Numerator $= 0 = \frac{\partial \vec{x}}{\partial s}\cdot\frac{\partial \vec{x}}{\partial t} = 0$. So endpoint $\frac{\partial \vec{x}}{\partial s}\perp\frac{\partial \vec{x}}{\partial t}$.

So at endpoint, either:
1. $\partial\vec{x}/\partial t \perp$ string or
2. $\partial\vec{x}/\partial t = 0$

But (2) can't be, because:

$$\vec{\mathcal{P}^{\sigma}} = T_0\sqrt{1 - v^2/c^2}\partial\vec{x}/\partial s = 0$$
$$v^2 = c^2 \text{ since } \partial\vec{x}/\partial s \neq 0$$

Motion perpendicular to string always if free.