Lecture 14 - Topics

- Momentum charges for the string
- Lorentz charges for the strings
- Angular momentum of the rotating string
- Discuss α' and the string length ℓ_s
- General gauges: Fixing τ and natural units

Reading: Section 8.4-8.6 and 9.1

$$S = \int d\xi^0 d\xi^1 \dots d\xi^p \mathcal{L}(\phi^a, \partial_\alpha \phi^a)$$

ξ^α: coordinates, $\phi^a(\xi)$: fields, $\partial_\alpha = \frac{\partial}{\partial \xi^\alpha}$

α: coord. index $\alpha = 0, 1, \dots, p$
a: field index $a = 1, \dots, m$
i: index for various symmetries

$$\delta\phi^a(\xi) =^i h_i^a(\phi(\xi))$$

Leaves $\mathcal{L}$ invar. to first order.
1.

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi^a}\delta\phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\partial_\alpha(\delta\phi^a) = 0}$$

3.

$$J^\mu \to J_i^\alpha \to \boxed{{}^i J_i^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)}\delta\phi^a}$$

Similar to mechanics: $\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q$

Claim: Given this transformation leaves $\mathcal{L}$ invar. to first order then:
2.

$$\boxed{\partial_\alpha J_i^\alpha = 0} \forall i \text{ Conserved Current}$$

Check this yourself using 1. and the $E - L$ equations of motion. Done in book as well.

Conserved charge too:

$$Q_i = \int J_i^O(\xi) d\xi^1 d\xi^2 \dots d\xi^p$$

Answer independent of time.

$$\boxed{\frac{dQ}{d\xi^O} = 0}$$

Nambu-Gotta action:

$$S = \int \underbrace{d\xi^0}_{d\tau} \underbrace{d\xi^1}_{d\sigma} \mathcal{L}(\partial_0 x^\mu, \partial_1 x^\mu)$$

This means $\alpha = 0, 1$. $\phi^a = x^\mu \Rightarrow a = 0, \ldots, d =$ spatial dimension
Let's look for asymmetry. A variation of the field that leaves the field invar.

$$\delta x^\mu =^\mu = \text{constant}$$

Constant translations of a worldsheet should by asymmetric. Why would Nambu-Gotta action care if rigidally moved worldsheet through time or space?
So:

$$\delta(\partial_0 x^\mu) = \partial_0(\delta x^\mu) = 0$$
$$\delta(\partial_1 x^\mu) = 0$$

So $\delta x^\mu =^\mu$ indeed asymmetric.

Apply (3)

$$^\mu J^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha x^\mu)}{}^\mu$$
$$J^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha x^\mu)}$$
$$(J^0_\mu, J^1_\mu) = \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \frac{\partial \mathcal{L}}{\partial x'^\mu}\right) = (\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu)$$

Conservation law: $\partial_\alpha J^\alpha = 0$ gives us collection of conservation laws for μ.

$$\partial_\alpha J^\alpha = 0 = \frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^a_\mu}{\partial a}$$

$P_\mu(\tau) = \int_0^{\sigma_1} \mathcal{P}^\tau_\mu(\tau, \sigma) d\sigma$ conserved quantity indexed by spacetime index μ.
$P_\mu(\tau)$: conserved momentum for the string not dependent on τ since conserved.

Check P_μ is conserved

$$\begin{aligned} \frac{dP_\mu}{d\tau} &= \int_0^{\sigma_1} \frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau}(\tau, \sigma) d\sigma \\ &= -\int_0^{\sigma_1} \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} d\sigma \\ &= [-\mathcal{P}^\sigma_\mu]_0^{\sigma_1} \end{aligned}$$

This yields the free BCs.

This is the hardest part of the course. After this, it gets easier.

A *momentum* is in general a variation of a Lagrangian with respect to a velocity eg $\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$ conserved, has units of momentum. We will see this is indeed the relative momentum of a piece of string.

When we had $(\rho, \vec{J})$:
$[\rho] = \frac{Q}{L^3}$
$[\vec{J}] = \frac{Q}{TL^2}$

Now we have $(\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu)$:
$[\mathcal{P}^\tau_\mu] = \frac{P_\mu}{L}$
$[\mathcal{P}^\sigma_\mu] = \frac{P_\mu}{T}$
Call $\mathcal{P}^\tau_\mu$ momentum density, and $\mathcal{P}^\sigma_\mu$ momentum current.

Okay, we have:

$$\frac{dP_\mu}{d\tau} = 0$$

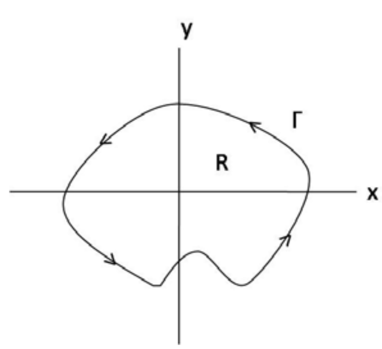
But would like:

$$\frac{dP_\mu}{dt} = 0$$

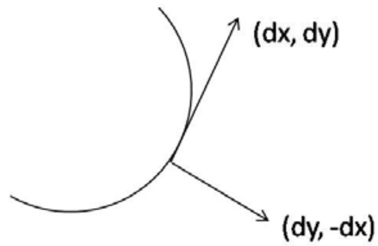
Conserved for Lorentz observer. Is this the case? (Yes).
Sure, could work in static gauge. $\tau = t \Rightarrow \frac{dP_\mu}{dt} = 0$

But what about an arbitrary τ curve on worldsheet?
Look for a generalization formula (clue from divergence theorem)

$A =$flux of vector field



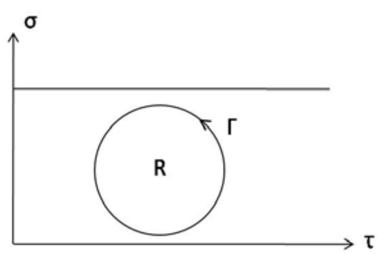
$$\oint (A^x dy - A^y dx) = \int_R \left(\frac{\partial A^x}{\partial x} + \frac{\partial A^y}{\partial y} \right) dx dy$$



$$\oint_\Gamma [\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau] =_R \left(\frac{\partial \mathcal{P}^\tau}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma}{\partial \sigma} \right) d\tau d\sigma = 0$$

Given an arbitrary curve γ, claim momentum given by:

$$\boxed{P_\mu(\gamma) = \int_\gamma [\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau]}$$



$$\gamma = \alpha \to \gamma_2 \to \beta \to -\gamma_1$$

$$\left(\int_{\gamma_2} + \int_\alpha + \int_{-\gamma_1} + \int_\beta\right) \underbrace{(\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau)}_{\kappa} = 0$$

$$\int_\alpha \kappa = \int_\beta \kappa = 0$$

$$P_\mu(\gamma_1) = P_\mu(\gamma_2)$$

Usually will use $P_\mu(\tau) = \int_0^{\sigma_1} \mathcal{P}^\tau_\mu(\tau, \sigma) d\tau$ with constant τ, but nice to have this general formulation.

Lorentz Transformation

$$x^\mu = L^\mu_\nu x^\nu$$

Leaves $\eta_{\mu\nu} x^\mu x^\nu$ invar. Vary x^μ subject to x^ν

$$\boxed{\delta x^\mu =^{\mu\alpha} x_\alpha}$$

$$\begin{aligned}\delta(\eta_{\mu\nu} x^\mu x^\nu) &= 2\eta^{\mu\alpha}_{\mu\nu} x_\alpha x^\nu \\ &= 2^{\mu\alpha} x_\alpha x_\mu\end{aligned}$$

where $\eta_{\mu\nu} x^\nu = x_\mu$

If we want $\delta(\eta_{\mu\nu} x^\mu x^\nu) = 0$, we make antisymmetric.

$$\boxed{^{\mu\nu} = -^{\nu\mu}}$$

Claim:

$$\delta(\eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu) = 0$$

So Nambu-Gotta action invar and get new set of symmetries:

$$\delta x^\mu(\tau, \sigma) =^{\mu\nu} x_\nu(\tau, \sigma)$$

$$^{\mu\nu} J^\alpha_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha x^\mu)} \delta x^\mu = \mathcal{P}^\alpha_\mu \delta x^\mu = \mathcal{P}^{\alpha\mu\nu}_\mu x_\nu$$

$$^{\mu\nu} J^\alpha_{\mu\nu} = -\frac{1}{2}^{\mu\nu} (x_\mu \mathcal{P}^\alpha_\nu - x_\nu \mathcal{P}^\alpha_\mu)$$

No physical relevance to $-\frac{1}{2}$
So define:

$$m^\alpha_{\mu\nu}(\tau, \sigma) = x_\mu \mathcal{P}^\alpha_\nu - x_\nu \mathcal{P}^\alpha_\mu$$

Conserved currents: $\partial_\alpha m^\alpha_{\mu\nu} = 0$

$$M_{\mu\nu} = \int_0^{\sigma_1} m^0_{\mu\nu} d\sigma$$

Conserved Charge

$$M_{ij} = \int_0^\sigma m_{ij} d\sigma = \int_0^{\sigma_1} (x_i \mathcal{P}^\tau_j - x_j \mathcal{P}^\tau_i) d\sigma =_{ijk} L_k$$

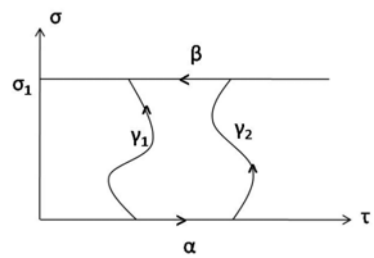
$_{123} = +1$, totally antisymmetric. eg:

$$M_{12} = \int_0^{\sigma_1} (x_1 \mathcal{P}_2^\tau - x_2 \mathcal{P}_1^\tau) d\sigma =_{12l} L_k = L_3$$

$$\vec{L} = \vec{r} \times \vec{p}$$

So M_{ij} =angular momentum, conserved.

Angular momentum of rotating string:



$$M_{12} = L_3 = J = \int_0^{\sigma_1} (x_1 \mathcal{P}_2^\tau - x_2 \mathcal{P}_1^\tau) d\sigma$$

$$\vec{x}(t,\sigma) = \frac{\sigma_1}{\pi} \cos\left(\frac{\pi\sigma}{\sigma_1}\right)\left(\cos\left(\frac{\pi ct}{\sigma_1}\right), \sin\left(\frac{\pi ct}{\sigma_1}\right)\right)$$

Parametrized String

$$\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\frac{\partial \vec{x}}{\partial t} = \frac{T_0}{c} \cos\frac{\pi\sigma}{\sigma_1}\left(-\sin\left(\frac{\pi ct}{\sigma_1}\right), \cos\left(\frac{\pi ct}{\sigma_1}\right)\right)$$

$$x_1 P_2 - x_2 P_1 = \left(\frac{\sigma_1}{\pi}\right)^2 \frac{T_0}{c} \cos^2\left(\frac{\pi\sigma}{\sigma_1}\right)$$

$$J = \frac{1}{2\pi T_0 c} E^2$$

$$\frac{E}{T_0} = \sigma_1$$