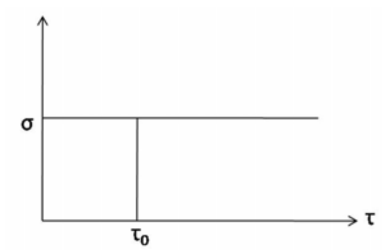
Lecture 15 - Topics

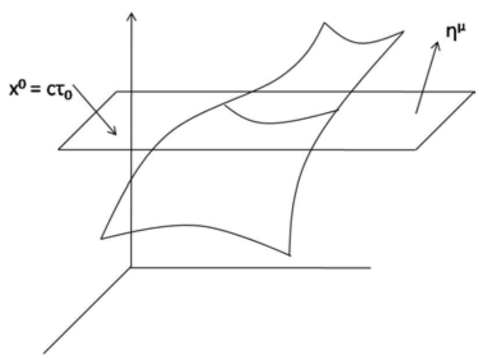
- Solution of the open string motion in the light-cone gauge

Reading: Sections 9.2-9.4

$$x^0(\tau_0, \sigma) = c\tau_0$$

$\tau = \tau_0$ is a line, goes to intersection of the worldsheet with the $x = c\tau_0$ hyperplane.





$$n_\mu x^\mu = \lambda\tau_0$$
$$n_\mu x_1^\mu = \lambda\tau_0$$
$$n_\mu x_2^\mu = \lambda\tau_0$$
$$n_\mu(x_1^\mu - x_2^\mu) = 0$$

$n_\mu = (1, \vec{0})$, $\lambda = c$, recover static gauge

If n^μ to be timelike:

$$n^\mu \Delta x_\mu = 0$$

Same for $\eta^\mu = (a, \vec{0})$. $\Delta x_\mu = (0, \vec{v})$.

Set λ usefully (aim at τ, σ dimensionless)

$$\boxed{n \cdot x(\tau,\sigma) = \widetilde{\lambda}\underbrace{(n \cdot p)}_{\text{const}}\tau}$$

$$P^{\mu} = \int_0^{\sigma_1} \mathcal{P}^{\tau\mu} d\sigma$$

$$\frac{dP^{\mu}}{d\tau} = -\mathcal{P}^{\sigma\mu}\Big]_0^{\sigma_1}$$

Ask for $n \cdot \mathcal{P}^{\sigma} = 0$ at endpoints so that $\frac{d}{d\tau}(n \cdot p) = 0$.

Reminder of Units:

J: Angular momentum of rotating string

$$\frac{J}{\hbar} = \alpha' E^2$$

$[\alpha'] = \frac{1}{[E]^2}$ since $\frac{J}{\hbar}$ is dimensionless.

Let's use *natural units* (as opposed to Planck units), set $c = 1, \hbar = 1$. Thus:

$$\frac{L}{T} = 1$$

$$ML^2 = 1 \Rightarrow \boxed{ML = 1}$$

Thus everything can be written in terms of units of length (sometimes people use mass instead)

So in natural units, $[\alpha'] = \frac{1}{[E]^2} = \frac{1}{M^2} = L^2$. So string length $l_s = \sqrt{\alpha'}$ in natural units (to get actual numbers, must replace the c's and $\hbar$'s)

$$l_s = \hbar c \sqrt{\alpha'}$$

In natural units:

$$\frac{T_0}{c} = \frac{1}{2}\pi\alpha'$$

To remember:

$$\boxed{l_s = \sqrt{\alpha'}}$$

$$\boxed{\frac{T_0}{c} = \frac{1}{2}\pi\alpha'}$$

Back to I. $L = [\widetilde{\ }]\frac{1}{L} \Rightarrow [\widetilde{\lambda}] = L^2 \Rightarrow \widetilde{\lambda} \propto \alpha'$.

As it turns out, $n \cdot x = 2\alpha'(n \cdot p)\tau$ (the 2 will be convenient)

σ parameterization

Static gauge:

$$\begin{aligned}\mathcal{P}^{\tau o} &= \frac{T_0}{c}\frac{(x')^2\dot{x}^o}{\sqrt{\cdots}} \\ &= \frac{T_0}{c}\frac{(\partial\vec{x}/\partial\sigma)^2}{ds/d\sigma\sqrt{1-v_\perp^2/c^2}}\end{aligned}$$

$$\left(\frac{\partial\vec{x}}{\partial\sigma}\right)^2 = \left(\frac{ds}{d\sigma}\right)^2$$

$$\boxed{\mathcal{P}^{\tau 0} = \frac{T_0}{c}\frac{ds/d\sigma}{\sqrt{1-v_\perp^2/c^2}}}$$

1. Try to make $n\cdot\mathcal{P}^\tau$ constant along the parameterized string.
2. Get a range $\sigma\in[0,\pi]$

Imagine had some parameter $\widetilde{\sigma}$,$\widetilde{\mathcal{P}}^{\tau\mu}(\tau,\sigma)$. If change parameter, how does it transform?

Claim transformation law:

$$\mathcal{P}^{\tau\mu}(\tau,\sigma) = \frac{d\widetilde{\sigma}}{d\sigma}\widetilde{\mathcal{P}}^{\tau\mu}(\tau,\widetilde{\sigma})$$

Makes sense that $\mathcal{P}^{\tau\mu}d\sigma$ is reparam. invar.

Multiply by n:

$$n\cdot\mathcal{P}^\tau(\tau,\sigma) = \frac{d\widetilde{\sigma}}{d\sigma}n\cdot\widetilde{\mathcal{P}}(\tau,\widetilde{\sigma})$$

Can set to be A is constant with respect to σ, might be a function of τ

$$\int_0^{\sigma_1} n\cdot\mathcal{P}^\tau(\tau,\sigma) = \sigma_1 A(\tau)$$

Also $\int = n\cdot P$ (momentum). So, $A(\tau) = n\cdot p/\sigma$. A not τ dependent!

$$n\cdot\mathcal{P}^\tau(\tau,\sigma) = \frac{n\cdot p}{\sigma_1}$$

$$\boxed{n\cdot\mathcal{P}^\tau(\tau,\sigma) = \frac{n\cdot p}{\pi}}$$

$\sigma\in[0,\pi]$

Recall eq. of motion of string:

$$\frac{\partial \mathcal{P}^{\sigma\mu}}{\partial \tau} + \frac{\partial \mathcal{P}^{\sigma\mu}}{\partial \sigma} = 0$$

Dot with n_μ

$$\frac{\partial}{\partial \tau}(n \cdot \mathcal{P}^\tau) + \frac{\partial}{\partial \sigma}(n \cdot \mathcal{P}^\sigma) = 0$$

$$\frac{\partial}{\partial \sigma}(n \cdot \mathcal{P}^\sigma) = 0$$

We had $n \cdot \mathcal{P}^\sigma = 0$ at string boundaries ($\sigma = 0, \sigma = \sigma_1 = \pi$) and since $\frac{\partial}{\partial \sigma}(n \cdot \mathcal{P}^\sigma) = 0$, $\boxed{n \cdot \mathcal{P}^\sigma = 0}$ $\forall \sigma$ and $\forall$ times.

Closed Strings

$n \cdot x = 2\alpha'(n \cdot p)\tau$

For closed strings, more convenient to remove 2:

$$\boxed{n \cdot x = \alpha'(n \cdot p)\tau}$$

$$\boxed{n \cdot \mathcal{P}^\tau(\tau, \sigma) = \frac{n \cdot p}{2\pi}}$$

$n \cdot \mathcal{P}^\sigma = 0$??? Do have $\frac{\partial}{\partial \sigma}(n \cdot \mathcal{P}^\sigma) = 0$, but don't have endpoints having $n \cdot \mathcal{P}^\sigma = 0$.

Open strings give rise to E&M. Closed strings give rise to gravity (harder! more subtle).

For a closed string, know how to put σ param. on strings at different times, but we don't know how to correlate these "σ ticks". No special points on closed string like endpoints on open string.

Compute:

$$n \cdot \mathcal{P}^\sigma = \frac{1}{2\pi\alpha'} \frac{(\dot{x} \cdot x') - \dot{x}^2 \partial_\sigma \widetilde{(n \cdot x)}}{\sqrt{\cdots}}$$

$n \cdot x \propto \tau$, so $\partial_\sigma(n \cdot x) = 0$, so to get $n \cdot \mathcal{P}^\sigma = 0$, make $\dot{x} \cdot x' = 0$. So in spacetime sense want $\dot{x} \perp x'$.

x': tangent to string
$\dot{x}$: line of constant σ

If given space vector x' and $\exists$ timelike vector, then $\exists$ unique vector orthogonal to x' prop to $\dot{x}$. So we lock the params. on the string, but still remains the ambiguity of translation of string (where do we set $\sigma = 0$ *No one knows.*)

Summary

1.

$$n \cdot x = \beta\alpha'(n \cdot p)\tau$$

where $\beta = 2$ if open string, or 1 if closed string.

2.

$$n \cdot \mathcal{P}^\tau = \frac{np\beta}{2\pi}$$

3.

$$\sigma \in [0, \frac{2\pi}{\beta}]$$

4.

$$n \cdot \mathcal{P}^\sigma = 0 \text{ everywhere } \Rightarrow \boxed{\dot{x} \cdot x' = 0}$$

$$\mathcal{P}^\tau_\mu = \frac{1}{2\pi\alpha'} \frac{x'^2 \dot{x}_\mu}{\sqrt{-\dot{x}^2 x'^2}}$$

Dot by n

$$n \cdot \mathcal{P}^\tau = \frac{1}{2\pi\alpha} \frac{x'^2}{\sqrt{-\dot{x}^2 x'^2}} \beta\alpha'(n \cdot p) = \frac{(n \cdot p)\beta}{2\pi}$$

$$1 = \frac{x'^2}{-\dot{x}^2 x'^2}$$

$$(x'^2)^2 = -(\dot{x}^2)(x'^2), \qquad (x')^2 \neq 0$$

$$x'^2 = \dot{x}^2$$

$$\boxed{\dot{x}^2 + x'^2 = 0}$$

Using (4), get:

$$\boxed{(\dot{x} \pm x')^2 = 0}$$

In static gauge, got:

$$\left(\frac{\partial \vec{x}}{\partial \sigma} \pm \frac{1}{c}\frac{\partial \vec{x}}{\partial t}\right) = 1$$

$$\boxed{\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \frac{\partial x^\mu}{\partial \tau}}$$

$$\boxed{\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'}\frac{\partial x^\mu}{\partial \sigma}}$$

Eq. of Motion:

$$\boxed{\ddot{x}^\mu - x^{\mu\prime\prime} = 0}$$

Wave equation for everyone!