Lecture 17 - Topics

- Light-cone fields and particles (cont'd.)

Reading: Sections 10.2-10.4

What are we doing now: Preparing grounds to see what arises from the string. How are particles described: Begin with simplest particle/field: the scalar field.

Lagrangian density for a scalar field $\phi(x)$:

$$\mathcal{L} = \frac{1}{2}(\partial_0\phi)^2 - \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}M^2\phi^2\right]$$

The first term represents the KE density and the second term represents the PE density.

Note since KE density has same units as PE density:

$$\left[\frac{1}{2}(\partial_0\phi)^2\right] = \left[\frac{1}{2}M^2\phi^2\right] \Rightarrow [M]$$

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2$$

$$S = \int d\vec{x}dt\mathcal{L}$$

$$E = \int Hd\vec{x} = \int d\vec{x}(\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}M^2\phi^2)$$

$$\begin{aligned}\delta S &= \int d\vec{x}dt(-\eta^{\mu\nu}\partial_\mu(\delta\phi)_\nu\phi - M^2\phi\delta\phi)\\ &= \int d\vec{x}dt\delta\phi(\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - M^2\phi)\end{aligned}$$

$$\boxed{(\partial^2 - M^2)\phi = 0}$$

$$\boxed{-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi - M^2\phi = 0}$$

This is the equation of motion of scalard field.

Next: Develop notion of scalar particles. How do we recognize them?

Plane Waves

Set scalar field to something that could satisfy equation of motion. Try:

$$\phi = a \exp(-iEt + i\vec{p} \cdot \vec{x})$$

Then:

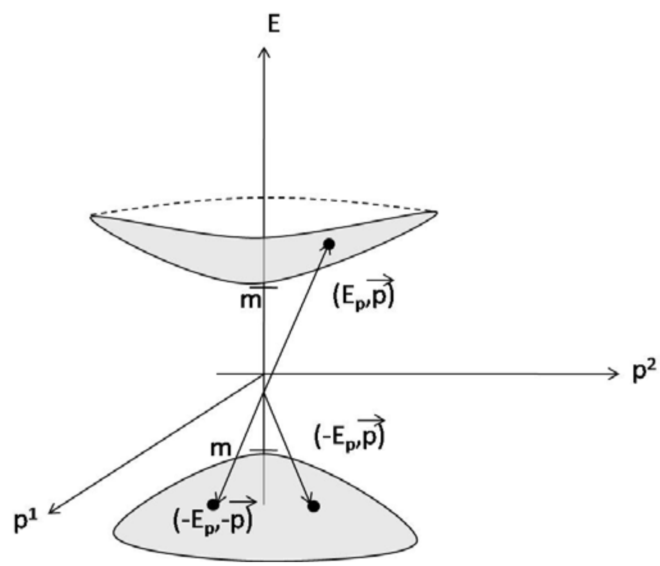
$$-(-iE)^2 + (i\vec{p}) \cdot (i\vec{p}) - M^2 = 0$$
$$E^2 - \vec{p}^2 = M^2 \Rightarrow -p^2 = M^2 \qquad (\text{ where } p = p_\mu p^\mu)$$

This looks sort of like a particle in quantum mechanics, but a bit naive. Try:

$$\phi = a \exp(-iEt + i\vec{p} \cdot \vec{x}) + a^* \exp(iEt - i\vec{p} \cdot \vec{x})$$

Can't anymore think of a particle with momentum p and energy E since get negative E. So abandon that interpretation.

Quantum Field Theory: The fields are dynamical variables and operations.



$$\phi(x) = \int \frac{d^P p}{(2\pi)^D} \exp(ip \cdot x)\phi(p)$$

$$(\phi(x))^* = \int \frac{d^p p}{(2\pi)^D} \exp(-ip \cdot x)(\phi(p))^* = \int \frac{d^P \vec{p}}{(2\pi)^D} \exp(ip \cdot x)(\phi(-p))^*$$

$$(\phi(x))^* = \int \frac{d^p \vec{p}}{(2\pi)^D} \exp(ip \cdot x)(\phi(p))$$

$$[\phi(p)]^* = \phi(-p)$$

If know value of field for some $(E_p, \vec{p})$

So geometrically, the reality condition of a point $(E_p, \vec{p})$ in momentum space in the top hyperboloid is equal to the realty condition of the complex conjugate in the bottom hyperboloid.

$$(\partial^2 - M^2)\int \frac{d^D p}{(2\pi)^D} \exp{(ip\cdot x)}\phi(p) = 0$$

$$\int \frac{d^D p}{(2\pi)^D}(-p^2 - M^2)\phi(p)\exp{(ipx)} = 0$$

$$\boxed{(p^2 + M^2)\phi(p) = 0}\,\forall p$$

Say $p^2 + M^2 \neq 0$ then $\phi(p) = 0$

Say $p^2 + M^2 = 0$ then $\phi(p)$ is arbitary.

This is the complete solution. A little simple sounding, but beautiful geometric interpretation. If not on hyperboloid, field vanishes. If on hyperboloid, field arbitrary (subject to reality condition).

$$\phi(p) \text{ determines } \phi(-p) = (\phi(p))^*$$

1 degree of freedom in the scalar field. (2 real numbers for two points).

Field Configuration

$$\phi_p(t, \vec{x}) = \frac{1}{\sqrt{v}}\frac{1}{\sqrt{2E_p}}(a(t)e^{i\vec{p}\cdot\vec{x}} + a^*(t)e^{-i\vec{p}\cdot\vec{x}})$$

$$V = L_1 L_2 L_3 \ldots L_d$$

$$x^i \approx x^i + L^i$$

$$p_i(x_i + L_i) = p_i x_i + 2\pi n_i$$

$$\boxed{p_i L_i = 2\pi n_i}$$

$$S = \int d\vec{x}dt(-\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}M^2\phi^2)$$

Can evaluate. Can do x integral, but cannot do t integral since t still arbitrary.

$$E = \int d\vec{x} H$$

$$S = \int dt \left(\frac{1}{2E_p} \dot{a}^*(t) a(t) - \frac{1}{2} E_p a^*(t) a(t) \right) \tag{1}$$

$$E = \frac{1}{2E_p} \dot{a}^*(t) \dot{a}(t) + \frac{1}{2} E_p a^*(t) a(t) \tag{2}$$

$$a(t) = q_1(t) + i q_2(t)$$

Thus:

$$S = \sum_{i=1}^{2} \int dt \left(\frac{1}{2E_p} \dot{q}_i^2 - \frac{1}{2} E_p q_i^2 \right)$$

This is a harmonic oscillator.

$$p_i = \frac{\partial S}{\partial q_i} = \frac{\dot{q}_i}{E_p}$$

$$p_1 + i p_2 = \frac{1}{E_p}(\dot{q}_1 + i \dot{q}_2) = \frac{\dot{a}(t)}{E_p}$$

Equation of motion:

$$\ddot{q}_i = -E_p^2 q_i$$

$$\boxed{\ddot{a}(t) = -E_p^2 a(t)}$$

$$a(t) = a_p e^{-iE_p t} + a_{-p}^* e^{iE_p t}$$

No reality condition is needed.

$$E = H = E_p (a_p^* a_p + a_{-p}^* a_{-p})$$

Let $a_{\vec{p}}$, $a_{-\vec{p}}$ be destruction operations. Let $a_{\vec{p}}^* \to a_{\vec{p}}^+$,$a_{-\vec{p}}^* \to a_{-\vec{p}}^+$ be creation operations.

$$[a_p, a_p^+] = 1 = [a_{-p}, a_{-p}^+]$$

All other commutators $= 0$.

How do we check this is okay?

$$[q_i(t), p_j(t)] = i\delta_{ij}$$

$$E = H = E_p(a_p^+ a_p + a_{-p}^+ a_{-p})$$

$$\phi_p(t, \vec{x}) = \frac{1}{\sqrt{v}} \frac{1}{\sqrt{2E_p}} (a(t)e^{i\vec{p}\cdot\vec{x}} + a^*(t)e^{-i\vec{p}\cdot\vec{x}})$$
$$= \frac{1}{\sqrt{v}} \frac{1}{\sqrt{2E_p}} (a_p e^{-iE_p t + ipx} + a_{-p} e^{iE_p t + ipx} + a_{p+} e^{iE_p t - i\vec{p}\cdot\vec{x}} + a_{-p} e^{iE_p t - i\vec{p}\vec{x}})$$

$$\boxed{\phi_p(t, \vec{x}) = \frac{1}{\sqrt{v}} \sum_{\vec{p}} \frac{1}{\sqrt{2E_p}} a_p e^{-iE_p t + i(\vec{p}\cdot\vec{x})} + a_p^+ e^{iE_p t - i(\vec{p}\cdot\vec{x})}}$$

$$\boxed{E = H = \sum_{\vec{p}} E_p a_{\vec{p}}^+ a_{\vec{p}}}$$

$$[a_{\vec{p}}, a^+ \vec{q}] = \delta_{\vec{p}, \vec{q}}$$

Define a vacuum state $|\Omega\rangle$:

$$a_{\vec{p}} |\Omega\rangle = 0 \forall \vec{p}$$
$$E |\Omega\rangle = 0$$

Create a state $a_{\vec{p}}^+ |\Omega\rangle$

Momentum Operator: $\vec{P} = \sum_{\vec{p}} \vec{p} a_p^+ a_p$. Note $\vec{P} |\Omega\rangle = 0$

$$\sum_{\vec{q}} = E_q a_q^+ a_q a_{\vec{p}} |\Omega\rangle = \sum_q E_q a_q^+ [a_q, a^+_{\vec{}}] |\Omega\rangle = E_{\vec{p}}(a_p^+ |\Omega\rangle)$$

So call $a_{\vec{p}} |\Omega\rangle$ a scalar particle of mass M, momentum $\vec{p}$, and energy $E_{\vec{p}} = \sqrt{\vec{p}^2 + M^2}$

Call a 1-particle state $a_{\vec{p_1}}^+, a_{\vec{p_2}}^+, \ldots, a_{\vec{p_n}}^+ |\Omega\rangle = n-$ particle state of total energy $E_{\vec{p_1}} + E_{\vec{p_2}} + \ldots + E_{\vec{p_n}}$ and momentum $\vec{p_1} + \vec{p_2} + \ldots + \vec{p_n}$

$$(E, p^1, p^2, \ldots, p^d) \leftrightarrow (p^+, p^-, p^I)$$

We have labelled the oscillators by the spatial components of the momentum which determine the energy.

Light-cone oscillators:

$$p^- = \frac{1}{2p^+}(p^{I^2} + M^2)$$