Lecture 19 - Topics

- Critical Dimension
- Constructing the State Space
- Tachyons

In the middle of quantizing the open string:

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}\frac{1}{p^+}L_n^\perp}$$

$$p^- = \frac{1}{2p^+\alpha'}(L_0^\perp + a)$$

What is a? a is arbitrary for now.

$$H = 2p^+p^-\alpha' = L_0^\perp + a$$

$$L_0^\perp = \alpha' p^I p^I + N^\perp$$

$$N^\perp = \sum_{k=1}^{\infty} k \underbrace{a_k^{I+} a_k^I}_{\text{Implicit sum over } I} \qquad \text{(Number operator)}$$

$$[N^\perp, a_\eta^{J+}] = n a_\eta^{J+}$$

$$M^2 = -p^2 = 2p^+p^- - p^I p^I = \frac{1}{\alpha'}(L_)^\perp + a) - p^I p^I = \frac{1}{\alpha'}(a + N^\perp)$$

$$[L_m^\perp, L_n^\perp] = (m-n)L_{m+n}^\perp + \frac{D-2}{12}m(m^2-1)\delta_{m+n,0}$$

With a lie algebra:

$$[A_1[B_1 C]] + [B_1[C_1 A]] + [C_1[A_1 B]] = 0$$

$$[L_m^+, X^I(\tau,\sigma)] = \xi_m^\tau \dot{X}^I + \zeta_m^\sigma X'^I$$

$$\zeta_m^\tau(\tau,\sigma) = -ie^{(im\tau)}\cos(m\sigma)$$

$$\zeta_m^\sigma(\tau,\sigma) = e^{(im\tau)}\sin(n\sigma)$$

$$\zeta_0^\sigma = 0, \zeta_0^\tau = -i$$

$$[L_0^\perp, X^I(\tau,\sigma)] = -i\dot{X}^I$$
$$i\partial_\tau \zeta = [\zeta, H]$$

Reparameterization of worldsheet:

$$X^I(\tau + \zeta_m^\tau, \sigma + \zeta_m^\sigma) = X^I(\tau,\sigma) + \underbrace{\zeta_m^\tau \dot{X}^I + \zeta_m^\sigma X'^I}_{[L_m^\perp, X^I]}$$

Critical Dimension

For a long time, wrote equations assuming $D = 4$. But came across problems in the quantum theory. Lovelace found some problems went away when $D = 26$ (considered a bad joke then)

Lorentz Charge:

$$\begin{aligned} M^{\mu\nu} &= \int_0^\pi d\sigma (X_\mu \mathcal{P}_\mu^\tau - X_\mu \mathcal{P}_\mu^\tau) \\ &= \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma (X_\mu \dot{X}_\nu - X_\nu \dot{X}_\mu) \\ &= X_0^\mu p^\nu - X_0^\nu p^\mu - i \sum_{n=1}^\infty \frac{1}{n} (\alpha_{-n}^\mu \alpha_n^\mu - \alpha_{-n}^\nu \alpha_n^\mu) \end{aligned}$$

Calculation sketched in book.

$$[M^{-I}, M^{-I}] = 0$$

$$M^{-I} = M^{\mu\nu}|_{\mu=-,\nu=I} = \underbrace{X_0^- p^I}_{\text{Hermitian}} - \underbrace{X_0^I p^-}_{\text{Not Hermitian}} \underbrace{- i \sum_{n=1}^\infty \frac{1}{n} (\alpha_{-n}^- \alpha_n^I - \alpha_{-n}^I \alpha_n^-)}_{\text{Hermitian}}$$

Rearrange so everything Hermitian:

$$M^{-I} = X_0^- p^I - \frac{1}{4p^+\alpha'}(X_0^I(L_0^\perp + a) + (L_0^\perp + a)X_0^I) - \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(L_{-n}^\perp \alpha_n^I - \alpha_{-n}^I L_n^\perp)$$

$$\boxed{[M^{-I}, M^{-I}] = \alpha' p^{+2}\sum_{m=1}^{\infty}(\alpha_{-m}^I\alpha_m^J - \alpha_{-m}^J\alpha_m^I)\cdot \underbrace{(m(1 - \frac{1}{24}(D-2)))}_{\text{“A”}} + \underbrace{\frac{1}{m}(\frac{1}{24}(D-2) + a)}_{\text{“B”}}}$$

$$mA + \frac{1}{m}B = 0 \qquad m = 1, 2, 3, 4, 5\ldots$$

$$m = 1 : A + B = 0$$
$$m = 2 : 2A + \frac{1}{2}B = 0$$

$A = 0, B = 0$:

$$\frac{D-2}{24} = 1 \Rightarrow \boxed{D = 26}$$
$$\frac{(26-2)}{24} + a = 0 \Rightarrow \boxed{a = -1}$$

$$\boxed{H = 2p^+p^-\alpha' = L_0^\perp - 1}$$
$$\boxed{M^2 = \frac{1}{\alpha'}(-1 + N)^\perp}$$

Einstein's theory makes sense in any number of dimensions. String theory fixes the dimension. No one really knows whether this 26-dimensional theory has anything to do with the 10- or 11-dimensional theory. What is a Kaluza-Klein Tower of States?

Constructing the State Space

Operators: x_0^I, p^I, x_0^-, p^+, a_n^I, a_n^{I+}

$$\left|p^+, \vec{p}_\tau\right\rangle \text{(Ground states } \forall \text{ values of momenta)}$$

By definition, annihilated by a_n^I:

$$a_n^I \left|p^+, \vec{p}_\tau\right\rangle = 0 \qquad n = 1, 2, 3, \ldots$$

$$M^2 \left|p^+, \vec{p}_\tau\right\rangle = \frac{1}{\alpha'} \left|p^+, \vec{p}_\tau\right\rangle \text{Scalar field of } M^2 = \frac{1}{-\alpha'}$$

$$\left|p^+, \vec{p}_\tau\right\rangle \leftrightarrow \text{Scalar Field}$$

$$\left|p^+, p_\tau\right\rangle : \begin{pmatrix} a_1^{(2)+} & a_1^{(3)+} & \cdots & a_1^{(25)+} \\ a_2^{(2)+} & a_2^{(3)+} & \cdots & a_2^{(25)+} \\ \vdots & \vdots & \vdots & \vdots \\ a_n^{(2)+} & a_1^{(3)+} & \cdots & a_1^{(25)+} \end{pmatrix}$$

General basis state of the state space:

$$|\lambda\rangle = \prod_{n=1}^{\infty} \prod_{I=2}^{25} (a_n^{I+})^{\lambda_{n,I}} \left|p^+, \widetilde{p}_\tau\right\rangle$$

$\lambda_{n,I}$ integers ≥ 0
$\forall n \geq 1, I = 2, \ldots, 25.$

To make mathematicians happy, restrict to case where states have only finite number of creation operations acting on the ground states.

$\forall \, |\lambda\rangle \, \exists$ only finite number of $\lambda_{n,I} \neq 0$.

String Hilbert space = infinite-dimension vector space (spanned by infinite set of linearly-independet basis $|\lambda\rangle$'s)

String theory describes an infinite number of different particles.

$N^{\perp} = 1$:

$$a_{\perp}^{I+} \left|p^{+}, \vec{p}_{\tau}\right\rangle \stackrel{D-2\text{states}}{\rightarrow} a^{I+}_{p^{+},p_{\tau}} \left|\Omega\right\rangle$$

$$M^2 = \frac{1}{\alpha'}(-1 + \underbrace{N^{\perp}}_{=1}) = 0$$

One-photon massless states! (Only massless bosons are photons). Started with classical strings, quantized them, and out popped photons!

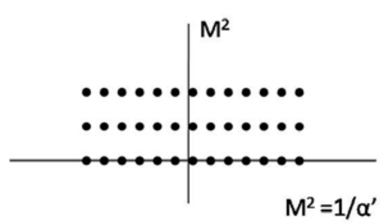
Will do the same with closed strings to get gravitons.

$N^{\perp} = 2$:

$$\underbrace{a_1^{I+}, a_1^{J+} \left|p^{+}, p_{\tau}\right\rangle}_{324 \text{ states}}, a_2^{I+} \left|p^{+}, p_{\tau}\right\rangle$$

Symmetric, traceless tensor in $D-1$ dimensions.

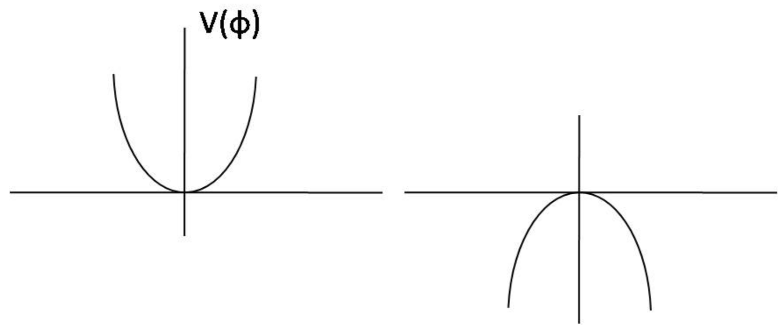
Tachyons:



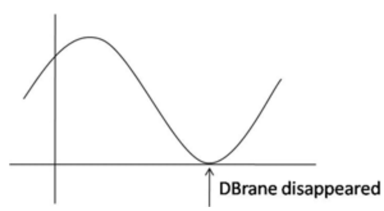
The more mass (up to the y axis), the less relevant it is to observed particle physics.

So might turn out that tachyons are the most relevant particles.
D-branes are made of tachyons! (Maybe)

$$V(\phi) = \frac{1}{2}M^2\phi^2 + \theta(\phi^3)$$



Instead of saying tachyons are massles, go faster than speed of light, etc.
Think of tachyons as an instability.
Instability in ... what? The D-branes (1999).
Too difficult to compute beyond T_{25}.



Tachyon Conjecture:

Zwiebach et al used computer simulations to get very high probability approximations.

Dec 2005: Analytic solution found. Energy exactly right. Tachyon conjecture verified: Tachyon instability is the instability of the D-brane and if it "rolls down" it destroys the D-brane.

D-brane collsions related to inflation? Maybe.