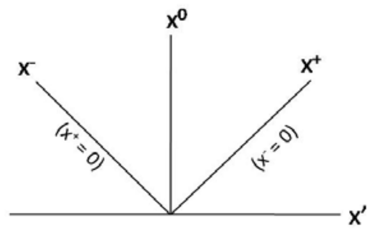
# Lecture 2 - Topics

- Energy and momentum
- Compact dimensions, orbifolds
- Quantum mechanics and the square well

Reading: Zwiebach, Sections: 2.4 - 2.9

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$$




## $x^+$ l.c. time

Leave $x^2$ and $x^3$ untouched.

$$\begin{aligned} -ds^2 &= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= \eta_{\mu v} dx^{\mu} dx^{v} \end{aligned}$$

$$u, v = 0, 1, 2, 3$$

$$\begin{aligned} 2dx^+ dx^- &= (dx^0 + dx^1)(dx^0 - dx^1) \\ &= (dx^0)^2 - (dx^1)^2 \end{aligned}$$

$$\begin{aligned} -ds^2 &= -2dx^+ dx^- + (dx^2)^2 + (dx^3)^2 \\ &= \hat{\eta}_{\mu v} dx^{\mu} dx^{v} \end{aligned}$$

$$u, v = +, -, 2, 3$$

$$\hat{\eta}_{\mu\nu} = \left[\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\hat{\eta}_{++} = \hat{\eta}_{--} = \hat{\eta}_{+I} = \hat{\eta} = -I$$

$$I = 2, 3$$

$$\hat{\eta}_{+-} = \hat{\eta}_{-+} = -1$$

$$\eta_{22} = \eta_{33} = 1$$

Given vector $a^\mu$, transform to:

$$a^\pm := \frac{1}{\sqrt{2}}(a^0 \pm a^1)$$

Einstein's equations in 3 space-time dimensions are great. But 2 dimensional space is not enough for life. Luckily, it works also in 4 dimensions (d5, d6, ...). Why don't we live with 4 space dimensions?
If we lived with 4 space dimesnions, planetary orbits wouldn't be stable (which would be a problem!)

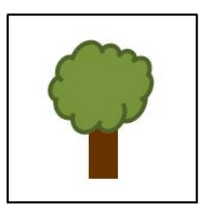
Maybe there's an extra dimension where we can unify gravity and ...

Maybe if so, then the extra dimensions would have to be very small – too small to see.

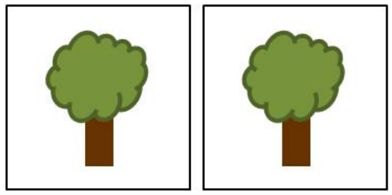
String theory has extra dimensions and makes theory work. Though caution: this *is* a pretty big leap.

## Trees in a Box

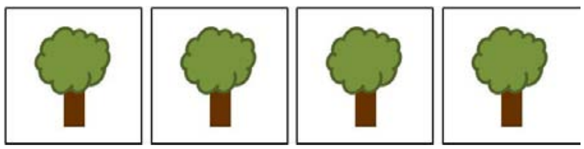
Look at trees in a box



Move a little and see another behind it



In fact, see $\infty$ row that are all identical! Leaves fall identically and everything.



# Dot Product

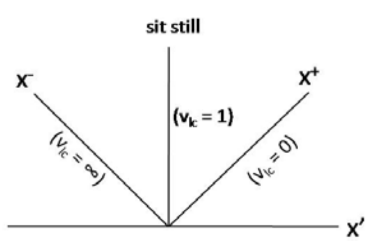
$$
\begin{aligned}
a \cdot b &= -a^{\circ} b^{\circ} + \sum_{i=1}^{3} a^i b^i \\
&= -a^+ b^- - a^- b^+ + a^2 b^2 + a^3 b^3 \\
&= \hat{\eta}_{\mu\nu} a^\mu b^\nu
\end{aligned}
$$

$$a_\mu = \hat{\eta}_{\mu\nu} a^\nu$$

$$a_+ = \hat{\eta}_{+\nu} a^\nu = \hat{\eta}_{+-} a^- = -a^-$$

$$\boxed{a_+ = -a^-}$$

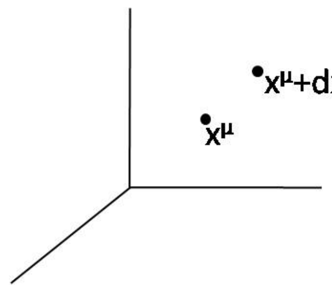
$$\boxed{a_- = -a^+}$$




$$v_{lc} = \frac{dx^-}{dx^+}$$

Light rays a bit like in Galilean physics - go from 0 to $\infty$.

## Energy and Momentum



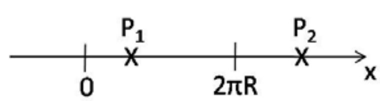

Event 1 at $x^\mu$
Event 2 at $x^\mu + dx^\mu$ (after some positive time change)

$dx^\mu$ is a Lorentz vector

The dimension along the room, row is actually a circle with one tree, so not actually infinity.

See light rayws that goes around circle multiple times to see multiple trees.

Crazy way to define a circle




This circle is a topological circle - no "center", no "radius"

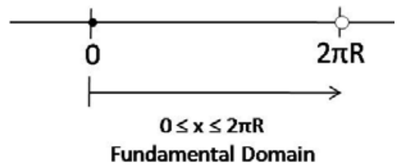
Identify two points, $P_1$ and $P_2$. Say the same ($P_1 \approx P_2$) if and only if $x(P_1) = x(P_2) + (2\pi R)n$ ($n \in Z$)

Write as:

$$\boxed{x \approx x + (2\pi R)n}$$

Define: Fundamental Domain = a region sit.
1. No two points in it are identified
2. Every point in the full space is either in the fundamental domain or has a representation in the fundamental domain.

So on our $x$ line, we would have:




$$
\begin{aligned}
-ds^2 &= -c^2dt^2 + (d\vec{x})^2 \\
&= -c^2dt^2 + v^2(dt)^2 \\
&= -c^2(1-\beta^2)(dt)^2
\end{aligned}
$$

$ds^2$ is a positive value so can take square root:

$$\boxed{ds = \sqrt{1-\beta^2}dt}$$

In to co-moving Lorentz frame, do same computation and find:

$$-ds^2 = -c^2(dt_p)^2 + (d\vec{x})^2 = -c^2(dt_p)^2$$

$dt_p$: Proper time moving with particle. Also greater than 0.

$$\boxed{ds = cdt_p}$$

$$\frac{dx^\mu}{ds} = \text{Lorentz Vector}$$

Define velocity u-vector:

$$u^\mu = \frac{cdcx^\mu}{dx}$$

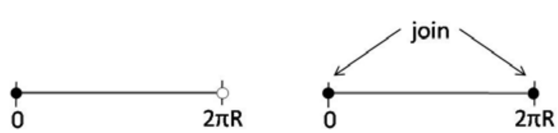
Definite momentum u-vector:

$$p^\mu = mu^\mu = \frac{m}{\sqrt{1-\beta^2}}\frac{dx^\mu}{dt} = m\gamma\frac{dx^\mu}{dt}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Rule to get the space we're trying to construct:

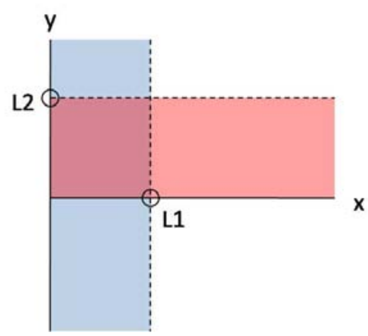
Take the $f \cdot d$, include its boundary, and apply the identification



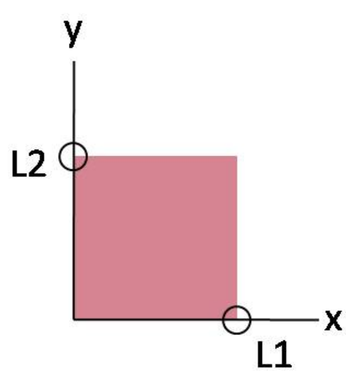
Note: Easy to get mixed up if rule not followed carefully.

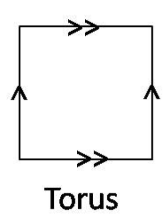
Consider $\Re^2$ with 2 identifications:

$$(x, y) \approx (x + L_1, y)$$
$$(x, y) \approx (x, y + L_2)$$




Blue: Fundamental domain for first identification
Red: Fundamental domain for second identification



$$\begin{aligned} p^\mu &= m\gamma\left(\frac{dx^0}{dt}, \frac{d\vec{x}}{dt}\right) \\ &= (mc\gamma, m\gamma\vec{v}) \\ &= \left(\frac{E}{c}, \vec{p}\right) \end{aligned}$$

$E$: relativistic energy $= \frac{\mu c^2}{\sqrt{1-\beta^2}}$
$\vec{p}$: relativistic momentum

Scalar:

$$\begin{aligned} p^\mu \cdot p_\mu &= (p^0)^2 + (\vec{p})^2 \\ &= -\frac{E^2}{c^2} + \vec{p}^2 \\ &= -\frac{m^2c^2}{1-\beta^2} + \frac{m^2v^2}{1-\beta^2} \\ &= -m^2c^2\left(\frac{1-\beta^2}{1-\beta^2}\right) \\ &= -m^2c^2 \end{aligned}$$

Every observer agrees on this value.

# Light Lone Energy

$x^0 =$ time, $\frac{E}{c} = p^0$
$x^+ =$ time, $\frac{E_{lc}}{c} = p^+$? –¿ Nope!

Justify using QM: $\Psi(t, \vec{x}) = e^{\frac{-i}{\hbar}(Et - \vec{p_0}\vec{x})}$

Can think of the IDs as transformations - points "move." Here's something that "moves" some points but not all.

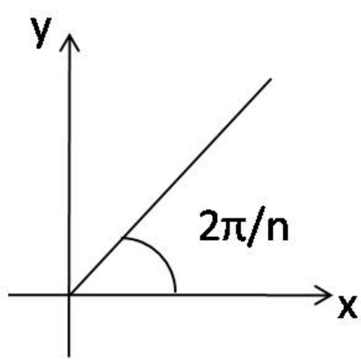
**Orbfolds**

1.


ID: $x \approx -x$
FD:


Think of ID as transformation $x \to -x$
This FD not a normal 1D manifold since origin is fixed. Call this half time $\Re/Z_z$ the quotient.

2.



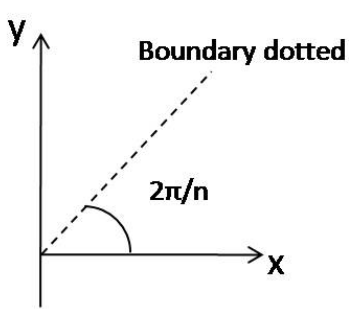

ID: $x \approx x$ rotated about origin by $2\pi/n$

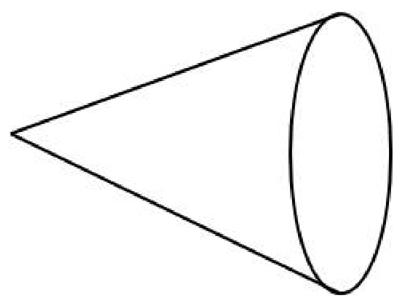
In polar coordinates:

$$z = x + iy$$

$$z \approx e^{\left(\frac{2\pi i}{n}\right)z}$$

Fundamental domain can be chosen to be:





Cone!

We focus on these two since quite solvable in string theory.

$$\hat{\vec{p}} = h\nabla/i$$

SE:

$$ih\frac{\partial\Psi}{\partial x^0} = \frac{E}{c}\Psi$$

$$\frac{ih}{c}\frac{\partial}{\partial t}\Psi = E\Psi$$

So for our $x^+$, want $ih\frac{\partial\Psi}{\partial x^+} = E_{lc}c\Psi$

$$\begin{aligned} Et - \vec{p}\cdot\vec{x} &= -\left(-\frac{E}{c}ct + \vec{p}\cdot\vec{x}\right) \\ &= -p\cdot x \\ &= -(p_+x^+p_-x^- + \ldots) \end{aligned}$$

Now have isolated dependence on $x^+$, so can take derivative:

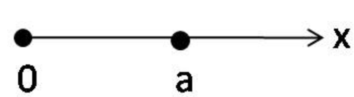
$$\Psi = e^{\frac{\pm i}{h}}(p_+x^+ + \ldots)$$

$$ih\frac{\partial\Psi}{\partial x^+} = -p_+\Psi$$

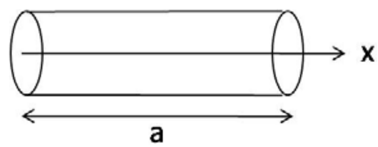
So:

$$\boxed{\frac{E_{lc}}{=} - p_+ = p^-}$$

Suppose have line segment of length $a$. Particle constrained to this:



Compare to physics of world with particle constrained to thin cylinder of radius $R$ and length $a$ (2D)




Can be defined as:


with ID $(x, y) \approx (x, y + 2\pi R)$

So:

$$SE = \frac{-h^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = E\Psi$$

1.

$$\Psi_k = \underbrace{}\sin\left(\frac{k\pi x}{a}\right)$$

$$E_k = \frac{h^2}{2m}\left(\frac{k\pi}{a}\right)^2$$

2.

$$\widetilde{\Psi}_{k,l} = \underbrace{}\sin\left(\frac{k\pi x}{a}\right)\cos\left(\frac{ly}{R}\right)$$

$$\Psi_{k,l} = \underbrace{}\sin\left(\frac{k\pi x}{a}\right)\sin\left(\frac{ly}{R}\right)$$

If states with $l = 0$ then get same states as case 1, but if $l \neq 0$ get different $E$ value from $\left(\frac{l}{R}\right)^2$ contribution. Only noticeable at very high temperatures.