Lecture 3 - Topics

- Relativistic electrodynamics.
- Gauss' law
- Gravitation and Planck's length

Reading: Zwiebach, Sections: 3.1 - 3.6

Electromagnetism and Relativity

Maxwell's Equations

Source-Free Equations:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{1}$$

$$\nabla \cdot \vec{B} = 0 \tag{2}$$

With Sources (Charge, Current):

$$\nabla \cdot \vec{E} = \rho \tag{3}$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{1}{c}\vec{J} \tag{4}$$

Notes:
1. E and B have same units.
3. ρ is charge density [charge/volume]. Here no $_0$ or 4π - those constants would get messy in higher dimensions.
4. $\vec{J}$ is current density [current/area]

$\vec{E}$, $\vec{B}$ are dynamical variables.

$$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right)$$

Solve the source free equations

$\nabla \cdot \vec{B} = 0$ solved by $\vec{B} = \nabla \times \vec{A}$. (Used to have $\nabla \times \vec{E} = 0, E = -\nabla\Phi$)

True equation:

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{A}) = \nabla \times \vec{E} + \frac{1}{c}\nabla \times \left(\frac{\partial \vec{A}}{\partial t}\right)$$
$$= \nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0$$

So:

$$\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\nabla\Phi \qquad (\Phi \text{ scalar})$$

Thus:

$$\boxed{\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}}$$

$(\vec{E}, \vec{B})$ encoded as $(\Phi, \vec{A})$
Φ, A are the fundamental quantities we'll use

Gauge Transformations

$$\vec{A} \to \vec{A}' = \vec{A} + \nabla$$
$$\vec{B}' = \nabla \times A' = \nabla \times (A + \nabla) = \vec{B}$$

function of $\vec{x}$,t. ∇ function = vector.

$$\Phi \to \Phi' = \Phi - \frac{1}{c}\frac{\partial}{\partial t}$$
$$\vec{E}' = -\nabla(\Phi') = -\nabla\left(\Phi - \frac{1}{c}\frac{\partial}{\partial t}\right) - \frac{1}{c}\frac{\partial}{\partial t}(A + \nabla) = \vec{E}$$

So under gauge transformations, $\vec{E}$ and $\vec{B}$ fields unchanged!

$$(\Phi, \vec{A}) \overset{\leftrightarrow}{g.t.} (\Phi', \vec{A}') \qquad \text{(Physically equivalent)}$$

Suppose 2 sets of potentials give the same $\vec{E}$'s and $\vec{B}$'s. Not guaranteed to be gauge-related.

Suppose we have 4-vector $A^\mu = (\Phi, \vec{A})$ then $A\mu = (-\Phi, \vec{A})$

Take $\frac{\partial}{\partial x^\mu}$. Have indices from $\frac{\partial}{\partial x^\mu}$ and from A^μ so will get a 4x4 matrix. Have two important quantities (E and B) with 3 components each $\Rightarrow$ 6 important quantities. Hint that we should get a symmetric matrix.

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{\mu\nu} = -F_{\nu\mu}$$

$$F_{oi} = \frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{\partial}{\partial x^i}(-\Phi) = -E_i$$

$$F_{12} = \partial_x A_y - \partial_y A_x = B_z$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

What happens under gauge transformation?

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu$$

Then get:

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\ &= \partial_\mu(A_\nu + \partial_\nu) - \partial_\nu(A_\mu + \partial_\mu) \\ &= F_{\mu\nu} + \partial_\mu\partial_\nu - \partial_\nu\partial_\mu \\ &= F_{\mu\nu} \end{aligned}$$

Define:

$$T_{\lambda\mu\nu} = \partial_x F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}$$

Note indices are cyclic.

Some interesting symmetries:

$$T_{\lambda\mu\nu} = -T_{\mu\lambda\nu}$$

$$T_{\lambda\mu\nu} = -T_{\lambda\nu\mu}$$

So $T_{\lambda\mu\nu}$ is totally antisymmetric. A totally symmetric object in 4D has only 4 nontrivial components so $T_{\lambda\mu\nu} = 0$ gives you 4 equations.

$$T_{\lambda\mu\nu} = 0 = \partial_\lambda(-\partial_\nu A_\mu) + \partial_\mu(\partial_\nu A_\lambda) + \partial_\nu(\partial_\lambda A_\mu - \partial_\mu A_\lambda)$$

Charge Q is a Lorentz invar. Not everything that is conserved is a Lorentz invar. eg. energy. Since Q is both conserved and a Lorentz invar, $(c\rho, \vec{J})$ form a 4-vector J^μ

Now let's do what a typical theoretical physicist does for a living: guess the equation!

$$F^{\mu\nu} \approx J^\mu \qquad \text{No, derivatives not right.}$$

$$\partial F^{\mu\nu}/\partial x^\nu \approx J^\mu \qquad \text{No, constants not right.}$$

$$\boxed{F^{\mu\nu}/\partial x^\mu = \frac{1}{c} J^\mu} \qquad \text{Correct, amazingly! (even sign)}$$

$\mu = 0$:

$$\partial F^{0\nu}/\partial x^\nu = \rho$$
$$\partial F^{0i}/\partial x^i = \rho$$
$$F_{0i} = -E_i$$
$$F^{0i} = E_i$$

So $\nabla \cdot \vec{E} = \rho$ verified!

Electromagnetism in a nutshell:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{J^\mu}{c}$$

Consider electromagnetism in 2D xy plane. Get rid of E_z component:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & B_z \\ E_y & -B_z & 0 \end{pmatrix}$$

But what about B_z? Doesn't push particle out of the plane ($v \times B_z$ with v in the xy plane remains in xy plane) but rename B_z as B, a scalar.

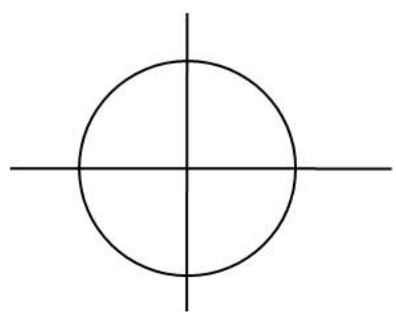
How about in 4D spatial dimensions?

$$F = \left(\begin{array}{c|cccc} & -E_x & -E_y & -E_z & -E_N \\ \hline & 0 & * & * & * \\ & & 0 & * & * \\ & & & 0 & * \\ & & & & 0 \end{array} \right)$$

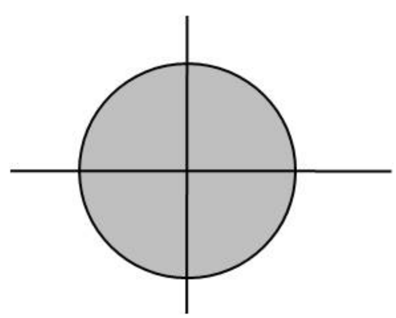
So get tensor B!

It's a coincidence that in our 3D spacial world E and B are both vectors.

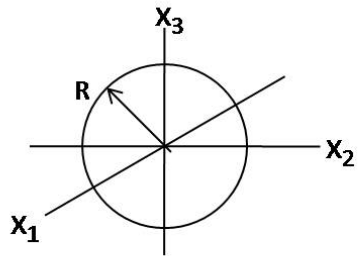
Let's look at $\nabla \cdot \vec{E} = \rho$ in all dimensions.



Notation: Circle S' is a 1D manifold, the boundary of a ball B^2



Sphere $S^2(R) : x_1^2 + x_2^2 + x_3^2 = R^2$
Ball $B^3(R) : x_1^2 + x_2^2 + x_3^2 \leq R^2$



When talking about $S^2(R)$, call it S^2 ($R = 1$ implied)
$\mathrm{Vol}(S^1) = 2\pi$
$\mathrm{Vol}(S^2) = 4\pi$
$\mathrm{Vol}(S^3) = 2\pi^2$

$$\mathrm{Vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

All you need to know about the Gamma function:

$$\Gamma(1/2) = \sqrt{\pi}$$

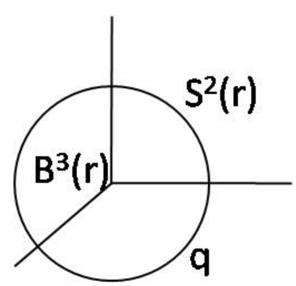
$$\Gamma(1) = 1$$

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(n) = (n-1)! \text{ for } n \in Z$$

$$\Gamma(x) = \int_0^\infty dt e^{-t} t^{x-1} \text{ for } x > 0$$

Calculating $\nabla \cdot \vec{E}$ in $d = 3$ and general d dimensions.
$d = 3$:



$$\int_{B^3(r)} \nabla \cdot \vec{E} d(\mathrm{vol}) = \int_{B^3(r)} \rho d(\mathrm{vol}) = q$$

This represents the flux of $\vec{E}$ through $S^2(r)$

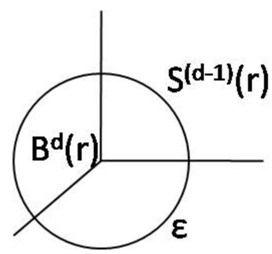
$$E(r) \cdot \mathrm{vol}(S^2(r)) = q$$

$$E(r) \cdot 4\pi r^2 = q$$

$$\boxed{E(r) = \frac{1}{4\pi}\frac{q}{r^2}}$$

This falls off much faster at large r and increases much faster as small r.

General d:



$$\int_{B^d(r)} \nabla \cdot \vec{E} d(\text{vol}) = \int_{B^d(r)} \rho d(\text{vol}) = q$$

This represents the flux of $\vec{E}$ through $S^{d-1}(r)$

$$E(r) \cdot \text{vol}(S^{d-1}(r)) = q$$

$$\boxed{E(r) = \frac{\Gamma(d/2)}{2\pi^{d/2}}\frac{q}{r^{d-1}}}$$

Electric field of a point charge in d dimensions.

If there are extra dimensions, then would see larger E at very small distances.