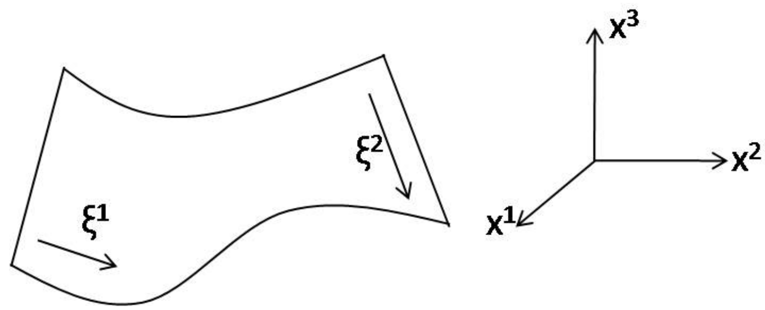
Lecture 7 - Topics

- Area formula for spacial surfaces

Area formula for spatial surfaces

("spatial" as opposed to "space-time")

Consider 2D surface in 3D space



3D Space

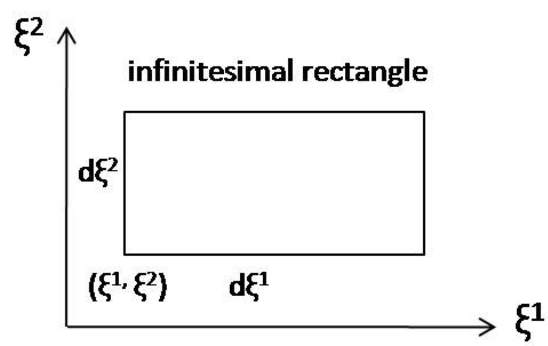
$$\vec{x} = (x^1, x^2, x^3)$$

Parameter Space: ξ^1, ξ^2 (directions along grid lines. Purely arbitrary. No connection to distances.)

Describe surface:

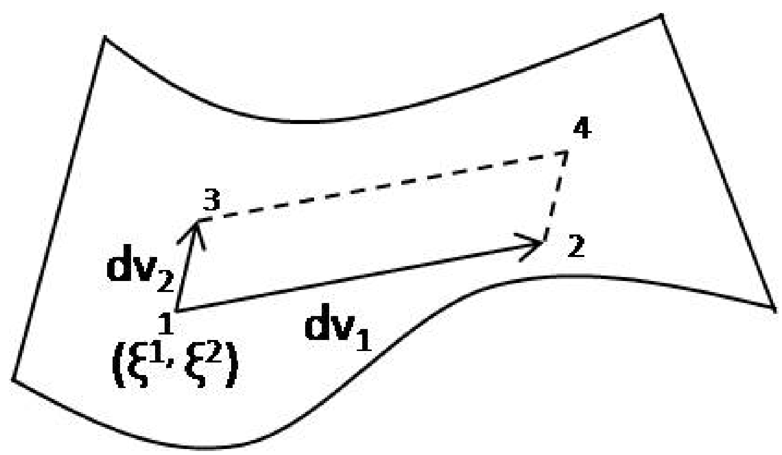
$$\vec{x}(\xi^1, \xi^2) = (x^1(\xi^1, \xi^2), x^2(\xi^1, \xi^2), x^3(\xi^1, \xi^2))$$

What is area, A?



$$A = \int_{\xi_1,\xi_2} \text{infinitesimal rectangles}$$

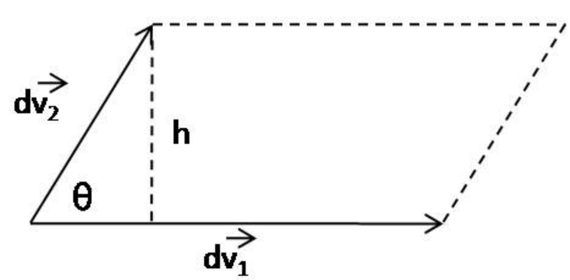
Map to surfae:



$d\vec{v}_1$: infinitesimal vector corresponding to $d\xi^1$ on ξ_1

To linear order, these 1, 2, 3, 4 points form a parallelogram

$d\vec{v}_1$: Mapping of bottom line of rectangle $= \frac{\partial \vec{x}}{\partial \xi^1}(\xi^1, \xi^2)d\xi^1$
$d\vec{v}_2$: Mapping of left line of rectangle $= \frac{\partial \vec{x}}{\partial \xi^2}(\xi^1, \xi^2)d\xi^2$



$$\begin{aligned} dA &= \text{base} \cdot \text{height} \\ &= |d\vec{v}_1| \cdot |d\vec{v}_2| \sin\theta \\ &= \sqrt{|dv_1|^2 |dv_2|^2 - (d\vec{v}_1 \cdot d\vec{v}^2)^2} \\ &= d\xi^1 d\xi^2 \sqrt{\left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2}\right)\left(\frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^2}\right) - \left(\frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2}\right)^2} \end{aligned}$$

$$A = \int dA$$

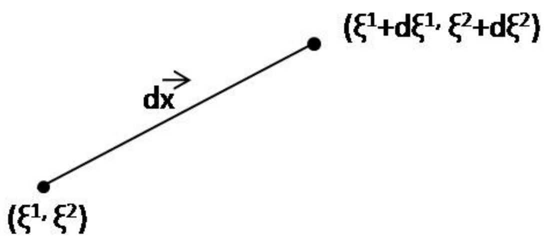
Important that this formula is reparameterization-invariant.

Reparam. Invariance

Choose another coordinate par. $(\widetilde{\xi}^1, \widetilde{\xi}^2)$. Can write as functions of our (ξ^1, ξ^2) coordinates. Must have:

$$dA = \int d\widetilde{\xi}^1 d\widetilde{\xi}^2 \sqrt{\left(\frac{\partial \vec{x}}{\partial \widetilde{\xi}^1} \cdot \frac{\partial \vec{x}}{\partial \widetilde{\xi}^2}\right)\left(\frac{\partial \vec{x}}{\partial \widetilde{\xi}^2} \cdot \frac{\partial \vec{x}}{\partial \widetilde{\xi}^2}\right) - \left(\frac{\partial \vec{x}}{\partial \widetilde{\xi}^1} \cdot \frac{\partial \vec{x}}{\partial \widetilde{\xi}^2}\right)^2}$$

Metric



$$d\vec{x} = \frac{\partial \vec{x}}{\partial \xi^1} d\xi^1 + \frac{\partial \vec{x}}{\partial \xi^2} d\xi^2 = \underbrace{\frac{\partial \vec{x}}{\partial \xi_i} d\xi^i}_{\text{implicit sum } i=1,2}$$

$$ds^2 = |d\vec{x}|^2 = d\vec{x} \cdot d\vec{x} = \underbrace{\frac{\partial \vec{x}}{\partial \xi^i} \frac{\partial \vec{x}}{\partial \xi^j}}_{\text{This is the metric.}} d\xi^i d\xi^j$$

$$= g_{ij}(\xi^1, \xi^2) d\xi^1 d\xi^2$$

Where metric $g_{ij} = \frac{\partial \vec{x}}{\partial \xi^i} \frac{\partial \vec{x}}{\partial \xi^j}$

Called the "induced metric" (induced because metric not made up but rather determined/inherited from the metric in the space the surface was embedded in).

$$g_{ij} = \begin{bmatrix} \frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^1} & \frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^1} \\ \frac{\partial \vec{x}}{\partial \xi^1} \cdot \frac{\partial \vec{x}}{\partial \xi^2} & \frac{\partial \vec{x}}{\partial \xi^2} \cdot \frac{\partial \vec{x}}{\partial \xi^2} \end{bmatrix}$$

$$A = \int d\xi^1 d\xi^2 \sqrt{g}$$

where $g = det(g_{ij})$