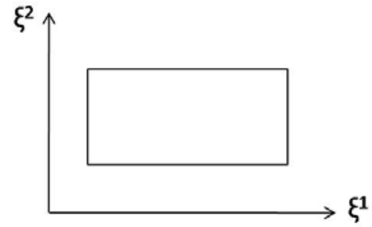
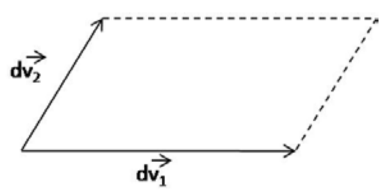
In general, $A = \int d\xi^1, d\xi^2 \sqrt{g}$



Note a line moving along in ξ^1 direction is not necessarily orthogonal to a line moving in ξ^2 direction.

$$d\vec{v}_1 = (d\xi^1, 0)$$
$$d\vec{v}_2 = (0, d\xi^2)$$

2D: no reference to 3D space, just 2D space param. by ξ^1 and ξ^2



$$dA = |d\vec{v}_1||d\vec{v}_2| \sin\theta$$
$$(d\vec{v}_1)^2 = g_{ij} dv_1^i dv_1^j = g_{11}(d\xi^1)^2$$
$$d\vec{v}_1 \cdot d\vec{v}_2 = g_{ij} dv_1^i dv_2^j = g_{12} d\xi^1 d\xi^2$$

$$\begin{aligned} dA &= \sqrt{[g_{11}(d\xi)^2][g_{22}(d\xi^2)^2] - [g_{12} d\xi^1 d\xi^2]^2} \\ &= d\xi^1 d\xi^2 \sqrt{g_{11} g_{22} - g_{12}^2} \\ &= d\xi^1 d\xi^2 \sqrt{det(g_{ij})} \end{aligned}$$

Works in any number of dimensions (though here proved only for 2)

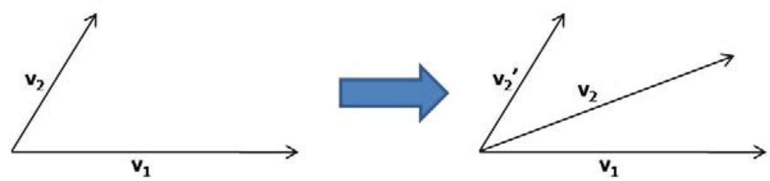
Generalization to n dimensions

Metric always a square matrix with a determinant.
Consider generalized parallelopiped in N dimensions. Volume in terms of corner vectors? (Standard from N-dim Euclidean geometry)

$$\boxed{\text{Vol} = det[V_i^k]}$$

where k is the vector index and i is the v_i subscripts $[1, \ldots, N]$

Can construct orthogonal vector sets



v_2' involves adding or subtracting as much of v_1 to v_2 to get orthogonality. Shifts parallelopiped into rectangle without changing volume. Every operation is determinant-invar.