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8.323 Relativistic Quantum Field Theory I Spring 2008

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## MASSACHUSETTS INSTITUTE OF TECHNOLOGY Physics Department

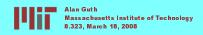
8.323: Relativistic Quantum Field Theory I

# Particle Creation by a Classical Source

(Part II, and incomplete)

March 18, 2008

— Alan Guth



#### Solution to Differential Equation

Equation of motion:

$$(\Box + m^2)\phi(x) = j(x) . \tag{2.1}$$

Initial condition:

$$\phi(x) = \phi_{\rm in}(x) \ . \tag{2.2}$$

Eqs. (2.1) and (2.2)  $\implies$  unique solution for Heisenberg operator  $\phi(x)$ .

Solution:

$$\phi(x) = \phi_{\text{in}}(x) + i \int d^4y \, D_R(x-y) j(y) ,$$
 (2.3)

where  $D_R(x-y)$  is the retarded propagator:

$$(\Box_x + m^2)D_R(x - y) = -i\delta^{(4)}(x - y)$$
  
where  $D_R(x - y) = 0$  if  $x^0 < y^0$  (retarded).



We know that

$$\begin{split} D_R(x-y) &= \theta(x^0 - y^0) \, \langle 0 \, | [\phi_{\text{in}}(x) \,, \, \phi_{\text{in}}(y)] | \, 0 \rangle \\ &= \theta(x^0 - y^0) \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \frac{1}{2E_p} \left[ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right]_{p^0 = E_p = \sqrt{\vec{p}^2 + m^2}} \\ &\qquad (2.5 \end{split}$$

Note that  $D_R(x-y)$  is defined by the <u>free</u> wave equation. It can be written in terms of  $[\phi_{\text{in}}(x), \phi_{\text{in}}(y)]$  as above, or in terms of  $[\phi_{\text{out}}(x), \phi_{\text{out}}(y)]$ , but <u>not</u> in terms of  $[\phi(x), \phi(y)]$ .

 $\theta(x^0-y^0)$  in  $D_R$  is hard to deal with, but for  $x^0\equiv t>t_2$  we can set  $\theta(x^0-y^0)=1$ . Then

$$\phi(x) = \phi_{\text{in}}(x) + i \int d^4 y \, j(y) \int \frac{d^3 p}{(2\pi)^3} \, \frac{1}{2E_p} \left[ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right] . \quad (2.6)$$

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Repeating,

$$\phi(x) = \phi_{\text{in}}(x) + i \int d^4 y \, j(y) \int \frac{d^3 p}{(2\pi)^3} \, \frac{1}{2E_p} \left[ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right] . \quad (2.6)$$

Define

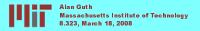
$$\tilde{\jmath}(p) \equiv \int d^4 y \, e^{ip \cdot y} \, j(y) ,$$
(2.7)

SO

$$\phi(x) = \phi_{\text{in}}(x) + i \int \frac{d^{3}p}{(2\pi)^{3}} \frac{1}{2E_{p}} \left[ \tilde{\jmath}(p)e^{-ip\cdot x} - \tilde{\jmath}(-p)e^{ip\cdot x} \right]$$

$$= \int \frac{d^{3}p}{(2\pi)^{3}} \frac{1}{\sqrt{2E_{p}}} \left\{ \left[ a_{\text{in}}(\vec{p}) + \frac{i}{\sqrt{2E_{p}}} \tilde{\jmath}(p) \right] e^{-ip\cdot x} + \left[ a_{\text{in}}^{\dagger}(\vec{p}) - \frac{i}{\sqrt{2E_{p}}} \tilde{\jmath}(-p) \right] e^{ip\cdot x} \right\}$$

$$= \int \frac{d^{3}p}{(2\pi)^{3}} \frac{1}{\sqrt{2E_{p}}} \left[ a_{\text{out}}(\vec{p})e^{-ipx} + \text{h.c.} \right] .$$
(2.8)



So

$$a_{\text{out}}(\vec{p}) = a_{\text{in}}(\vec{p}) + \frac{i}{\sqrt{2E_p}} \tilde{\jmath}(p)$$

$$a_{\text{out}}^{\dagger}(\vec{p}) = a_{\text{in}}^{\dagger}(\vec{p}) - \frac{i}{\sqrt{2E_p}} \tilde{\jmath}(-p) ,$$
(2.9)

where

$$\tilde{\jmath}(-p) = \tilde{\jmath}^*(p) , \qquad (2.10)$$

since j(x) is real, and

$$p^0 = \sqrt{\vec{p}^2 + m^2} \ . \tag{2.11}$$

Thus, only the mass shell component  $(p^0 = \sqrt{\vec{p}^2 + m^2})$  of  $\tilde{\jmath}(p)$  results in particle creation. This is just the classical phenomenon of resonance occurring in the quantum field theory setting.



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#### Unitary Transformation Between In and Out

It is useful to construct a unitary transformation that relates in and out quantities.

Remembering that  $D_R(x-y)=\theta(x^0-y^0)\,\langle 0\,|[\phi_{\rm in}(x)\,,\,\phi_{\rm in}(y)]|\,0\rangle$ , recall also that  $[\phi_{\rm in}(x)\,,\,\phi_{\rm in}(y)]$  is a c-number, so  $\langle 0\,|[\phi_{\rm in}(x)\,,\,\phi_{\rm in}(y)]|\,0\rangle=[\phi_{\rm in}(x)\,,\,\phi_{\rm in}(y)]$ . So for  $x^0\equiv t>t_2$ ,

$$\phi(x) = \phi_{\text{out}}(x) = \phi_{\text{in}}(x) + i \int d^4y \, \left[\phi_{\text{in}}(x), \, \phi_{\text{in}}(y)\right] \, j(y) \, .$$
 (2.12)

If we define

$$B \equiv \int d^4 y \, j(y) \, \phi_{\rm in}(y) , \qquad (2.13)$$

then

$$\phi_{\text{out}}(x) = \phi_{\text{in}}(x) + i \left[\phi_{\text{in}}(x), B\right] .$$
 (2.14)

But  $[\phi_{in}(x), B]$  is also a c-number, so we can write

$$\phi_{\text{out}}(x) = e^{-iB} \phi_{\text{in}}(x) e^{iB} . \tag{2.15}$$



Since

$$\phi_{\text{out}}(x) = e^{-iB} \phi_{\text{in}}(x) e^{iB} , \qquad (2.15)$$

we know from the uniqueness of the Fourier expansion that

$$a_{\text{out}}(\vec{p}) = e^{-iB} a_{\text{in}}(\vec{p})e^{iB}$$
 (2.16)

We can also verify that this equation is true by using

$$a_{\text{out}}(\vec{p}) = a_{\text{in}}(\vec{p}) + \frac{i}{\sqrt{2E_p}}\tilde{\jmath}(p)$$
 (2.9a)

with

$$\left[ a_{\rm in}(\vec{p}) \,,\, a_{\rm in}^{\dagger}(\vec{q}) \right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \;. \tag{2.17}$$



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#### The S-Matrix

Define

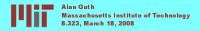
$$S \equiv e^{iB}$$
 (the **FAMOUS** S-Matrix) (2.18)

Mapping of states:

$$a_{\text{out}}(\vec{p}) |0_{\text{out}}\rangle = 0$$
 
$$S^{-1}a_{\text{in}}(\vec{p})S |0_{\text{out}}\rangle = 0$$
 
$$\implies a_{\text{in}}(\vec{p})S |0_{\text{out}}\rangle = 0$$
 (2.19)

This implies, up to a phase, the  $S |0_{\rm out}\rangle = |0_{\rm in}\rangle$ . We can redefine the phase of  $|0_{\rm out}\rangle$  (or  $|0_{\rm in}\rangle$ ) so that

$$S \left| 0_{\text{out}} \right\rangle = \left| 0_{\text{in}} \right\rangle . \tag{2.20}$$



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On one particle states,

$$S |\vec{p}_{\text{out}}\rangle = Sa_{\text{out}}^{\dagger}(\vec{p}) |0_{\text{out}}\rangle$$

$$= \underbrace{Sa_{\text{out}}^{\dagger}(\vec{p})S^{-1}}_{a_{\text{in}}^{\dagger}(\vec{p})} \underbrace{S|0_{\text{out}}\rangle}_{|0_{\text{in}}\rangle}$$

$$= |\vec{p}_{\text{in}}\rangle$$
(2.21)

In general, we could show that

$$S \left| \vec{p}_1 \dots \vec{p}_{N, \text{out}} \right\rangle = \left| \vec{p}_1 \dots \vec{p}_{N, \text{in}} \right\rangle$$
 (2.22)

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### Normal Ordering of S

We know that

$$S = e^{iB} = e^{i \int d^4 y \, j(y) \, \phi_{in}(y)}$$
 (2.23)

It is useful to write S so that all the annihilation operators are on the right. Let

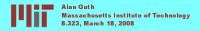
$$iB = i \int d^4 y \, j(y) \int \frac{d^3 p}{(2\pi)^3} \, \frac{1}{\sqrt{2E_p}} \left[ a_{\text{in}}(\vec{p}) e^{-ip \cdot y} + a_{\text{in}}^{\dagger}(\vec{p}) e^{ip \cdot y} \right] = G + F ,$$
(2.24)

where

$$F = i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \tilde{\jmath}(p) a_{\rm in}^{\dagger}(\vec{p}) , \quad G = i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \tilde{\jmath}(-p) a_{\rm in}(\vec{p}) , \quad (2.25)$$

where we recall that

$$\tilde{\jmath}(p) \equiv \int d^4 y \, e^{ip \cdot y} \, j(y) \ .$$
 (2.7)



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So

$$S = e^{iB} = e^{F+G} {.} {(2.26)}$$

F and G do not commute, but [F,G] is a c-number and therefore commutes with both F and G. Whenever F and G commute with [F,G],

$$e^{F+G} = e^F e^G e^{-\frac{1}{2}[F,G]} . {(2.27)}$$

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## Aside about $e^{F+G}=e^Fe^Ge^{-rac{1}{2}[F\,,G]}$

To prove this identity, define

$$H_1(\lambda) \equiv e^{\lambda(F+G)}$$
,  $H_2(\lambda) = e^{\lambda F} e^{\lambda G} e^{-\frac{1}{2}\lambda^2[F,G]}$ . (2.28)

Clearly  $H_1(0) = H_2(0) = I$  (identity operator), and

$$\frac{\mathsf{d}H_1(\lambda)}{\mathsf{d}\lambda} = (F+G)H_1(\lambda) \ . \tag{2.29}$$

So if we can show that  $H_2(\lambda)$  obeys the same differential equation as above, then it follows that  $H_2(\lambda) = H_1(\lambda)$ . You'll get to show this on your next problem set.

This is actually a special case of the Baker-Campbell-Hausdorff formula, which has the general form

$$e^F e^G = e^{F+G+\frac{1}{2}[F,G]+\dots\text{(iterated commutators)}}$$
 (2.30)

We'll prove this, too, on a problem set soon.

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#### Returning to the main argument:

So

$$S = e^{iB} = e^{F+G} {.} {(2.26)}$$

F and G do not commute, but  $[F\,,\,G]$  is a c-number and therefore commutes with both F and G. Whenever F and G commute with [F, G],

$$e^{F+G} = e^F e^G e^{-\frac{1}{2}[F,G]}$$
 (2.27)

Recalling

$$F = i \int \frac{\mathsf{d}^3 p}{(2\pi)^3} \, \frac{1}{\sqrt{2E_p}} \tilde{\jmath}(p) a_{\rm in}^{\dagger}(\vec{p}) \;, \quad G = i \int \frac{\mathsf{d}^3 p}{(2\pi)^3} \, \frac{1}{\sqrt{2E_p}} \tilde{\jmath}(-p) a_{\rm in}(\vec{p}) \;, \quad (2.25)$$

one sees that

e sees that 
$$[F, G] = -\int \frac{\mathsf{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{\mathsf{d}^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \tilde{\jmath}(p) \, \tilde{\jmath}(-q) \underbrace{\left[a_\mathsf{in}^\dagger(\vec{p}), a_\mathsf{in}(\vec{q})\right]}_{-(2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})} \qquad (2.31)$$

$$= \int \frac{\mathsf{d}^3 p}{(2\pi)^3} \, \frac{1}{2E_p} \, |\tilde{\jmath}(p)|^2 \quad .$$

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So

$$S = e^{iB} = \exp\left\{-\frac{1}{2} \int \frac{\mathsf{d}^3 p}{(2\pi)^3} \, \frac{1}{2E_p} \, |\tilde{\jmath}(p)|^2\right\} \, e^F \, e^G \quad . \tag{2.32}$$

### Vacuum to Vacuum Probability

The probability that no particles are produced by the source is given by

$$P(\text{no particle production}) = |\langle 0_{\text{out}} | 0_{\text{in}} \rangle|^2$$
 (2.33)

Logic: physical state is  $|0_{in}\rangle$ , independent of time (in the Heisenberg picture).

 $|0_{\text{out}}\rangle = \text{state}$  with no particles for  $t > t_2$ .

So,  $|\langle 0_{\rm out} | 0_{\rm in} \rangle|^2$  is the probability that the physical state of the system would be measured to have 0 particles at  $t > t_2$ . To express the answer in terms of the S-matrix, recall

$$|0_{\text{out}}\rangle = S^{-1}|0_{\text{in}}\rangle \implies \langle 0_{\text{out}}| = \langle 0_{\text{in}}|S.$$
 (2.34)

So

$$\begin{split} \langle \mathbf{0}_{\text{out}} \, | \mathbf{0}_{\text{in}} \rangle &= \langle \mathbf{0}_{\text{in}} \, | S | \, \mathbf{0}_{\text{in}} \rangle \\ &= \exp \left\{ -\frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \frac{1}{2E_p} \, | \tilde{\jmath}(p) |^2 \right\} \langle \mathbf{0}_{\text{in}} \, \left| e^F e^G \right| \mathbf{0}_{\text{in}} \rangle \ . \end{split} \tag{2.35}$$



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