

8.513 Problem Set 5

(Dated: October 15, 2004)

Due Oct 28

1. Consider a system of bosons interacting with each other through some generic two-body interaction $V(r)$. In this problem you will study various aspects of the superfluid state in some generality.
 - (a) Write the interaction in second quantized form. Use this to write down the action that enters the coherent state path integral. (Remember that the interaction is two-body, i.e. a boson at position r interacts with another at position r' through $V(r - r')$). For what follows it will be convenient to express the interaction in terms of the Fourier components of the boson density.
 - (b) Now assume that at the classical minimum the boson field has a non-zero expectation value ψ_0 . As was done in class for the specific case of a δ -function interaction, expand the action about the classical minimum to second order in the deviation of the density from its mean.
 - (c) Now integrate out the density fluctuation to obtain an effective quadratic action for the phase variable. (Hint: this is best done in Fourier space).
 - (d) Use the strategy of Prob 3 in Prob Set 4 to obtain a condition for when the superfluid state is stable at zero temperature. Your answer will depend on the fourier transform $V(q)$ of the interaction, and the spatial dimensionality. Specifically, calculate the expectation value $\langle e^{i\theta} \rangle$ using the quadratic phase action derived. Some of you might find it convenient (though not necessary) to regard it as a sum of harmonic oscillators - one for each momentum - and to use the known result for oscillators.
 - (e) If the superfluid is stable, what is the dispersion of the phase mode in terms of $V(q)$? Show that your answer reduces to the correct one in the two limiting cases of (a) δ function interaction and (b) Coulomb interaction in $d = 3$.

2. Lattice bosons

Consider a system of bosons on a lattice with the Hamiltonian

$$H = - \sum_{\langle ij \rangle} t \left(b_i^\dagger b_j + h.c \right) - \mu \sum_i n_i + U \sum_i n_i (n_i - 1) \quad (1)$$

Here i, j refer to the sites of a cubic lattice, b_i is the boson annihilation operator at site i and $n_i = b_i^\dagger b_i$ is the number of bosons at site i . Assume $t > 0, U > 0$. This is known as the boson Hubbard model.

- (a) Assume $t \gg U$. Find the classical ground state of the system, i.e. consider b_i to be classical and find the minimum energy state.
- (b) Consider the theory of small harmonic fluctuations about this classical ground state. Show that the excitations consist of a linear dispersing sound wave at long wavelengths, i.e. the structure is unchanged from the translation invariant superfluid.
- (c) Now specialize to the case where the chemical potential is such that there are an integer number of bosons per site on average. In the limit $U \gg t$ is the ground state a superfluid?

3. Flux quantization in superconductors

A superconductor may roughly be thought of as a superfluid of charged bosons (known as Cooper pairs) which are formed by pairing together two electrons. The phase of the Cooper pair creation operator ψ^\dagger may be treated classically in the superconducting state. Due to the charge carried by the Cooper pair, it interacts with electromagnetic fields. As a consequence, in contrast to a neutral superfluid, the Hamiltonian describing the system must be invariant under the local “gauge” transformation

$$\psi \rightarrow \psi e^{\frac{2ie\phi(\vec{r})}{\hbar c}}; \vec{A} \rightarrow \vec{A} + \vec{\nabla}\phi \quad (2)$$

where \vec{A} is the electromagnetic vector potential. c is the speed of light, and e is the charge of the electron. The cost in energy of a static long wavelength disturbance in the Cooper pair phase θ together with the accompanying vector potential \vec{A} is:

$$E = \int d^3x \frac{\rho_s}{2} \left(\hbar \nabla \theta + \frac{2e\vec{A}}{c} \right)^2 + \frac{1}{8\pi} (\vec{\nabla} \times \vec{A})^2 \quad (3)$$

The first term generalizes the analogous term in the superfluid to take into account the coupling to the magnetic field. Note that this term is invariant under the gauge transformation mentioned above. The second term is simply the usual expression for the energy stored in the magnetic field.

(i) By minimizing this expression, derive the equations satisfied by the functions θ, \vec{A} . Show that these equations are consistent with the identification of the physical current as

$$\vec{j} = e\rho_s \left(\hbar \nabla \theta + \frac{2e\vec{A}}{c} \right) \quad (4)$$

In particular, your equations must simply be

$$\vec{\nabla} \cdot \vec{j} = 0 \quad (5)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{4\pi\vec{j}}{c} \quad (6)$$

The first equation is the continuity equation expressing charge conservation (remember that we are considering only static configurations so that the time derivative of the charge density is zero). The second equation is simply Ampere's law. Note that this expression for the current again generalizes the analogous expression for the superfluid, and is gauge invariant.

(ii) In the ground state, we may set θ to be a constant. Show that the equation for \vec{A} becomes

$$\nabla^2 \vec{A} = \frac{\vec{A}}{\lambda^2} \quad (7)$$

and derive an expression for the length scale λ in terms of ρ_s . This equation implies that a weak magnetic field will penetrate into a superconductor only upto the length scale λ .

To get some feeling for this, consider a semi-infinite slab of the superconductor which occupies all of space for $x > 0$. Assume that for $x < 0$, there is a magnetic field $\vec{B} = (0, 0, B_0)$ pointing along the z -axis. Solve the equation above to get the profile of the magnetic field inside the superconductor for $x > 0$.

This is the explanation of the Meissner effect: if a superconductor is cooled in a small external magnetic field, it will expel the field from its interior.

(iii) Now we will consider the properties of vortex configurations in θ . For concreteness, consider a cylindrical sample with a hole running through the middle. Assume that the phase θ winds around by 2π on going clockwise once around a loop that encircles the hole, *i.e.*

$$\int_C d\vec{l} \cdot \vec{\nabla} \theta = -2\pi \quad (8)$$

where the integral is taken around the loop C . Deep inside the bulk of the superconductor, the current will be zero. Show that this implies that there is magnetic flux of $hc/2e$ that is associated with the vortex. The magnetic field will extend upto a distance of order λ from the inner edge of the cylinder. For a vortex with winding number m , the associated flux will be $\frac{mhc}{2e}$. Thus the flux through the hole of a thick superconducting cylinder is quantized to be an integer multiple of $hc/2e$.

Suggested reading: P.W. Anderson, *Basic notions of condensed matter physics*, pages 229-248.