Bardeen-Cooper-Schrieffer theory

1. Quasiparticles.

Consider quasiparticles of a BCS superconductor,

$$\mathcal{H} = \sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}} a^+_{\mathbf{p},\sigma} a_{\mathbf{p},\sigma} + \sum_{\mathbf{p}} \left(\bar{\Delta} a_{\mathbf{p},\uparrow} a_{-\mathbf{p},\downarrow} + h.c. \right) = \sum_{\mathbf{p},\sigma} E_{\mathbf{p}} b^+_{\mathbf{p},\sigma} b_{\mathbf{p},\sigma} \tag{1}$$

where $E_{\mathbf{p}} = \sqrt{\epsilon^2_{\mathbf{p}} + |\Delta|^2}$ and $b_{\mathbf{p},\uparrow} = u_{\mathbf{p}} a_{\mathbf{p},\uparrow} + v_{\mathbf{p}} a^+_{-\mathbf{p},\downarrow}$, etc., are Bogoliubov quasiparticle operators.

a) Find out how many particles are contained in one quasiparticle. For that, consider a state with one quasiparticle added to the BCS ground state,

$$|\mathbf{p}, \sigma\rangle = b^+_{\mathbf{p},\sigma} |0_{\rm BCS}\rangle \tag{2}$$

evaluate the expectation value

$$\langle N \rangle = \langle \mathbf{p}, \sigma | \sum_{\mathbf{q},\alpha} a^+_{\mathbf{q},\alpha} a_{\mathbf{q},\alpha} | \mathbf{p}, \sigma \rangle \tag{3}$$

and express the result in terms of the Bogoliubov angle $\theta_{\mathbf{p}}$. Can $\langle N \rangle$ be nagative? Explain.

b) Consider momentum and spin of a quasiparticle in the state (2). What are they? Do they depend on the Bogoliubov angle?

2. Gap equation.

For a BCS superconductor derive the gap equation

$$\Delta = \lambda \nu \int_{-E_*}^{E_*} u_{\mathbf{p}} \bar{v}_{\mathbf{p}} \tanh \left(\frac{1}{2} \beta E_{\mathbf{p}} \right) d\epsilon \tag{4}$$

with E_* the interaction cutoff parameter ($E_* \sim E_F$ for nonretarded contact interaction).

Study the gap Δ as a function of temperature. Show that Δ decreases monotonically with T and vanishes at a certain temperature $T = T_c$. Find the value T_c.

3. Gap suppression in a superflow. Critical current.

Superflow in a superconductor is described by the order parameter with spatially varying phase, $\Delta(\mathbf{r}) = \Delta e^{2i\mathbf{qr}}$, which is related to the superflow velocity by $\mathbf{v}_s = \mathbf{q}/m$.

BCS quasiparticles in the presence of superflow are described by the Hamiltonian

$$\mathcal{H} = \sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}} a^+_{\mathbf{p},\sigma} a_{\mathbf{p},\sigma} + \sum_{\mathbf{p}} \left(\bar{\Delta} a_{\mathbf{p+q},\uparrow} a_{-\mathbf{p+q},\downarrow} + h.c. \right) \tag{5}$$

which can be dioganalized by a Bogoliubov transformation in which the states $\mathbf{p} + \mathbf{q}$ and $-\mathbf{p} + \mathbf{q}$ are paired up.

a) Find the quasiparticle spectrum. Assuming $|\mathbf{q}| \ll p_F$, show that the result can be interpreted in terms of Doppler shift $E'_{\mathbf{p}} = E_{\mathbf{p}} + \mathbf{v}_s \mathbf{p}$, where $E_{\mathbf{p}}$ is the spectrum in the absence of the flow.

b) Show that the energy gap between the BCS ground state and the first excited state is reduced in the presence of the flow. Find the critical velocity at which the gap vanishes.

c) Consider BCS pairing in the frame co-moving with the flow. By using Galilean invariance, or otherwise, argue that the gap equation and thus the order parameter Δ are not affected by the flow. Combined with the result of part b), this shows that the energy gap and pairing amplitude Δ aint necessarily have to be equal. They happen to be equal in a clean superconductor in the absence of external pair-breaking fields or flows, but are not equal in general.