(released late due to instructor illness)

Path Integral

Reading: R. P. Feynman and A. P. Hibbs, Quantum Mechanics and Path Integrals; M. Stone, The Physics of Quantum Fields

1. Harmonic oscillator.

a) Starting from a path integral representation, obtain the Greens function of a harmonic oscillator:

$$\langle x', t | x, 0 \rangle = \left(\frac{m\omega}{2\pi i \sin \omega t} \right)^{1/2} \exp\left(\frac{im\omega}{2\sin\omega t} \left[(x^2 + x'^2)\cos\omega t - 2xx' \right] \right) \tag{1}$$

To find the prefactor, evaluate the infinite product of gaussian integrals using the formula

$$\frac{z}{\sin z} = \prod_{n>0} \left(1 - \frac{z^2}{\pi^2 n^2} \right)^{-1} \tag{2}$$

Note the periodicity in t and the singularities at $\omega t = \pi n$, $n \in Z$. Discuss their meaning and check that in the limit $\omega \to 0$ the Greens function of the Schrödinger equation for a free particle is recovered.

b) Consider the density matrix $\rho(x, x') = \langle x' | e^{-\beta \mathcal{H}} | x \rangle$ of a harmonic oscillator at finite temperature ($\beta = 1/k_{\rm B}T$). It is convenient to evaluate $\rho(x, x')$ using the imaginary time path integral representation, $\rho(x, x') = \int d[x(\tau)] e^{-S}$, with the integral take over the paths $x(\tau)_{0<\tau<\beta}$ constrained by $x(0) = x$, $x(\beta) = x'$. Find $\rho(x, x')$ by continuing the result of part a) to an imaginary time $t \to -i\beta$.

Evaluate the partition function as $Z = {\rm tr}\, e^{-\beta\mathcal{H}} = \int \rho(x, x) dx$ and check that the result agrees with the standard geometric series sum $Z = \sum_n e^{-\beta E_n}$, $E_n = \hbar\omega(n + \frac{1}{2})$.

2. Ising chain and the double well potential.

Consider a one-dimensional Ising problem with a model ferromagnetic interaction between the spins that falls exponentially:

$$\mathcal{H}[s_i] = -\frac{1}{2} \sum_{i,j} J_{ij} s_i s_j, \quad J_{ij} = J e^{-k|i-j|}, \quad s_i = \pm 1 \tag{3}$$

Using the Hubbard-Stratonovich transformation

$$e^{\frac{1}{2}\sum_{i,j} A_{ij} s_i s_j} = (2\pi)^{-N/2} \int ... \int \prod_{i=1}^{N} dh_i e^{-\frac{1}{2}\sum_{i,j} A^{-1}_{ij} h_i h_j - \sum_i h_i s_i} \tag{4}$$

bring the Ising partition function to the form

$$Z = \int ... \int \prod_{i=1}^{N} dh_i e^{-\frac{1}{2}\sum_{i,j} A^{-1}_{ij} h_i h_j + \sum_i \ln(2\cosh h_i)} \quad A_{ij} = \beta J e^{-k|i-j|} \tag{5}$$

Invert the matrix A_{ij} (Hint: use Fourier representation) and show that

$$A_{ij}^{-1} = \begin{cases} C, & i = j \\ B, & i = j \pm 1 \\ 0, & \text{else} \end{cases} \qquad B = \frac{T}{J \sinh k} \qquad C = \frac{T}{J} \coth k \tag{6}$$

Use this result to rewrite the partition function as an imaginary time path integral

$$Z = \int ... \int dh_1 ... dh_N \exp\left(- \sum_i \left[\frac{B}{2} (h_i - h_{i+1})^2 + U(h_i) \right] \right) \qquad U(h) = (C - B)h^2 - \ln(2 \cosh h) \tag{7}$$

Plot $U(h)$ for different temperatures. Find the 'phase transition' temperature T_c below which the potential U becomes a double well.

b) There is no ordering in the Ising model in 1D due to thermal fluctuations. The behavior at $T < T_c$ can be understood by comparing with the imaginary time path integral for a qunatum mechanical particle in a double well potential,

$$\int d[x(\tau)] \exp\left[- \int \left(\frac{m}{2} \dot{x}^2 + U(x) \right) d\tau \right] \tag{8}$$

(Argue that at the large radius of spin intercation, $k \ll 1$, the discrete variable i can be replaced by a continuous time variable.)

Discuss the meaning of varius statistical mechanical quantities in the quantum-mechanical instanton tunneling landuage. Relate the correlation length of magnetic ordering in the Ising chain with the tunnel even-odd level splitting. Use the instanton solution for the double well potential to estimate the correltion length at $T \ll T_c$.

c) Consider the above Ising model in the presence of an external magnetic field,

$$\mathcal{H}[s_i] = -\frac{1}{2} \sum_{i,j} J_{ij} s_i s_j - \sum_i B s_i \tag{9}$$

Find the magnetization $M = \langle s \rangle$ as a function of B. (Hint: use the tunneling/instanton picture and treat B as a double-well asymmetry parameter.)