# Lecture 10: QED I

0. Feynman rules for $\phi^3$ scalar theory
1. Schrödinger equation
2. Klein - Gordon equation
3. Dirac equation
4. Solutions to Dirac equation
5. Bilinear covariants

## 0. Feynman rules for $\phi^3$ scaler theory:

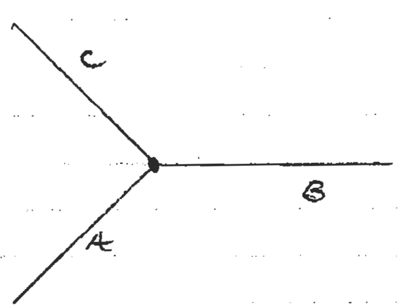
Introduce the concept of Feynman rules using a simpler approach then starting right away with QED where we have to deal with particles having a spin $\neq 0$.

We start with a scaler theory. Various conceptual ideas are introduced even in quantum field textbooks like e.g.: Lewis Ryder, Quantum Field Theory, which I strongly recommend!

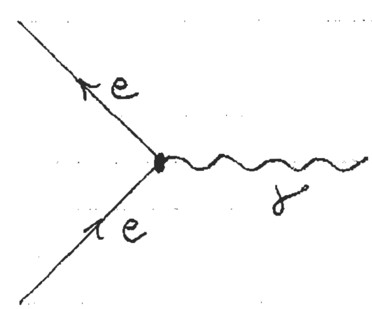
° Scalar theory: Lagrangian $\mathcal{L}$

$$\mathcal{L} = \underbrace{\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m \phi^2}_{\mathcal{L}_0 \text{ (free field)}} - \underbrace{g \phi^3}_{\mathcal{L}_{int} \text{ (interaction)}}$$

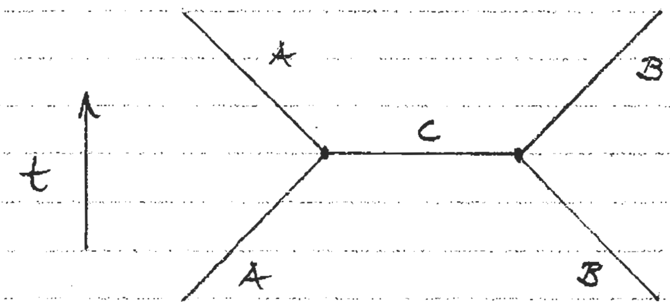
fundamental graph: have particles with different masses: A, B, C




compare this to:



- Scattering : $A + B \rightarrow A + B$




To get the cross-section applying Fermi's golden rule, we need:

- phase space integration
- Matrix element for underlying process

Recall, the matrix element quantifies the transition from an initial state $|i\rangle$ to a final state $|f\rangle$

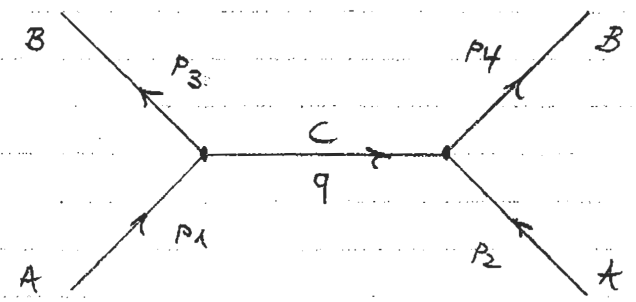
This amplitude can be obtained either using the

- canonical quantisation method or using the
- Path integral method

In both cases, a series expansion is performed with respect to the underlying coupling constant (perturbative evaluation).

1. Write down lowest Feynman diagram which reflects the lowest order term in a series expansion:

$A + A \rightarrow B + B$




- Feynman rules:

0. Draw graph for a particular process up to a certain order (here: lowest order)

1. Label the incoming and outgoing four-momenta $p_1, p_2, p_3, p_4$. Label internal lines: Here $q_1 = q$. Put an arrow on each line, to keep track of the positive direction: Notation

2. Coupling constant:

For each vertex, write down a factor of $-ig$

$g$: coupling constant here: $(-ig)^2$

3. Propagator:

For each internal line: $\dfrac{i}{q_j^2 - m_j^2 c^2}$

here: $\dfrac{i}{q^2 - m_C^2 c^2}$

4. Conservation of energy and momenta:

For each vertex, write down a delta function of the form:

$$(2\pi)^4 \delta^4(p_1 - p_3 - q) \qquad (2\pi)^4 \delta^4(p_2 + q - p_4)$$

Note: if the arrow leads outward, then use minus the four-momentum of that line!

5. Integration over internal momenta:

For each internal line: $\frac{1}{(2\pi)^4} d^4 q_j$

here: $\frac{1}{(2\pi)^4} d^4 q$ and integrate over all momenta!

Rule 1 - 5 together gives:

$$-i (2\pi)^4 g^2 \int \frac{1}{(q^2 - m_c^2 c^2)} (2\pi)^4 \delta^4(p_1 - p_3 - p_4) \delta^4(p_2 + q - p_4) d^4 q$$

Integration over q:

$$-i g^2 \frac{1}{(p_4 - p_2)^2 - m_c^2 c^2} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

6. Cancel the delta function:

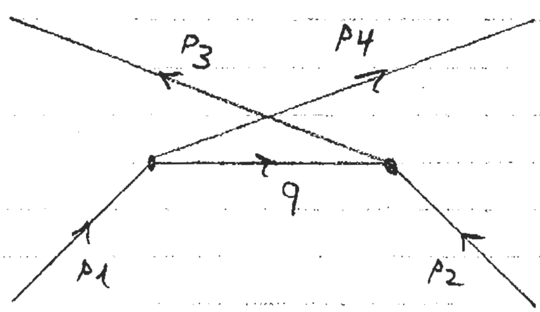
Erase: $(2\pi)^4 \delta^4(p_1 + p_2 + \dots - p_n)$

here: $(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$

What remains is: $-i\mathcal{M}$

$$\boxed{\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2 c^2}}$$

Now in our particular example there is another graph at lowest order: $A + A \to B + B$




amplitude $$\frac{g^2}{(p_3 - p_2)^2 - m_C^2 c^2}$$

- Total amplitude:

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2 c^2} + \frac{g^2}{(p_3 - p_2)^2 - m_C^2 c^2}$$

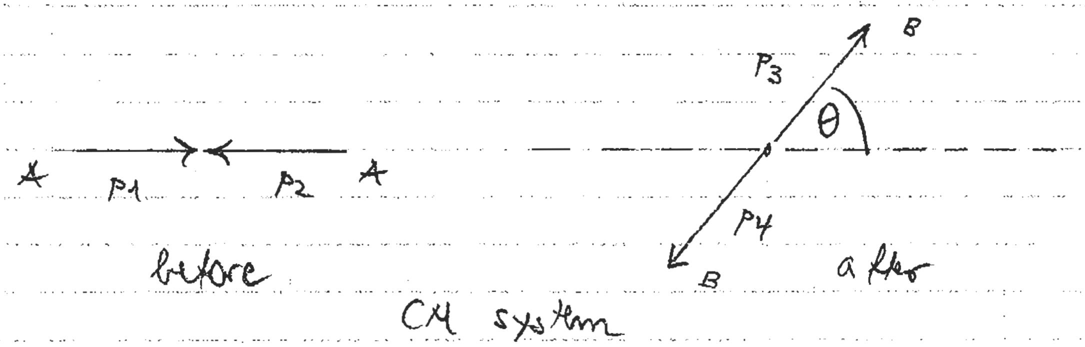
Simplification:

$m_A = m_B = m$ ; $m_C = 0$ (like the photon)

then:

$$(p_4 - p_2)^2 - m_C^2 c^2 = p_4^2 + p_2^2 - 2 p_2 \cdot p_4 = -2\vec{p}^{\,2}(1 - \cos\theta)$$

$$(p_3 - p_2)^2 - m_C^2 c^2 = p_3^2 + p_2^2 - 2 p_3 \cdot p_2 = -2\vec{p}^{\,2}(1 + \cos\theta)$$




$\vec{p}$ : incident momentum of particle 1

$$\boxed{\mathcal{M} = -\frac{g^2}{\vec{p}^{\,2}\sin^2\theta}} \longrightarrow \boxed{|\mathcal{M}|^2 = \frac{g^4}{\vec{p}^{\,4}\sin^4\theta}}$$

$\downarrow$

Determine: $\frac{d\sigma}{d\Omega} \propto |\mathcal{M}|^2$

1. Schrödinger equation:

classical approach:

$$\frac{\vec{p}^2}{2m} + V = E$$

go to operators:

$$\vec{p} \to \frac{\hbar}{i} \nabla \qquad E \to i\hbar \frac{\partial}{\partial t}$$

then we find: (acting on a wavefunction)

$$\boxed{-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = i\hbar \frac{\partial \psi}{\partial t}}$$

2. Klein-Gordon equation:

start: $E^2 - \vec{p}^2 c^2 = m^2 c^4$ or: $p^\mu p_\mu - m^2 c^2 = 0$

and now: $p_\mu \to i\hbar \partial_\mu$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

with: $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$ $\quad \partial_1 = \frac{\partial}{\partial x}$ $\quad \partial_2 = \frac{\partial}{\partial y}$ $\quad \partial_3 = \frac{\partial}{\partial z}$

(e get in terms of operators:

$$-\hbar^2 \partial^\mu \partial_\mu \psi - m^2 c^2 \psi = 0$$

and therefore: $-\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} + \nabla^2 \psi = \left(\frac{mc}{\hbar}\right)^2 \psi$

introduced d'Alembertian operator $\Box$:

$$\boxed{\Box \equiv \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2}$$

With that we can write: $+\hbar^2 \Box \psi + m^2 c^2 \psi = 0$

or: $\boxed{(\Box + m^2)\psi = 0}$ $\hbar = c = 1$

comments:

1. Schrödinger equation is first order in $t$
2. Klein-Gordon equation is second order in $t$

Klein-Gordon equation:

$\longrightarrow$ Problem with single-particle interpretation

However: Resolve in quantum field theory for a spin $\emptyset$ particle!

o Solutions: $\psi = e^{-i/\hbar\,(p_\mu x^\mu)} = e^{-i/\hbar\,(p_0 x^0 - \vec{p}\cdot\vec{x})} =$

$$= e^{i/\hbar\,(\vec{p}\cdot\vec{x} - Et)}$$

Insert this into the Klein-Gordon equation:

$$\boxed{E = \pm c\sqrt{m_0^2 c^2 + \vec{p}^{\,2}}}$$

2 solutions: - positive energy and

- negative energy

This was later interpreted as:

anti-particle: negative energy

particle: positive energy

note:

Relativistic quantum theory leads to new degrees of freedom: The charge degrees of freedom of a certain particle

o in case of Klein-Gordon: Spin $\phi$

## 3. Dirac equation

- Goal of Dirac: 1. Equation which is first order in $t$
  2. Equation which is consistent with the relativistic energy-momentum formula

Starting point:

1. Factor energy-momentum relation:

in $p^0$ only: $(p^0)^2 - m^2c^2 = (p^0 + mc)(p^0 - mc) = 0$

With that: Two first order solutions:

$$\boxed{(p^0 + mc) = 0 \quad \text{and} \quad (p^0 - mc) = 0}$$

How can we do this for: $p^\mu p_\mu - m^2c^2 = 0$

Ansatz:

$$(p^\mu p_\mu - m^2c^2) = (\beta^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc)$$

$\beta^\kappa, \gamma^\lambda$: eight "coefficients" to be determined now

$$\underline{\beta^\kappa \gamma^\lambda p_\kappa p_\lambda} - \underline{mc(\beta^\kappa - \gamma^\kappa) p_\kappa} - \underline{m^2 c^2} =$$

$$\stackrel{!}{=} (\underline{p^\mu p_\mu} - \underline{m^2 c^2})$$

<u>avoid terms linear in $p_\kappa$, so choose</u> : $\beta^\kappa = \gamma^\kappa$

<u>Then :</u> $\beta^\kappa \gamma^\lambda p_\kappa p_\lambda = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$

<u>calculate this now :</u> $p^\mu p_\mu = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$

$$(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 =$$

$$(\gamma^0)^2 (p^0)^2 + (\gamma^1)^2 (p^1)^2 + (\gamma^2)^2 (p^2)^2 + (\gamma^3)^2 (p^3)^2$$

$$\left.\begin{aligned}
&+ (\gamma^0\gamma^1 + \gamma^1\gamma^0)\, p_0 p_1 \\
&+ (\gamma^0\gamma^2 + \gamma^2\gamma^0)\, p_0 p_2 \\
&+ (\gamma^0\gamma^3 + \gamma^3\gamma^0)\, p_0 p_3 \\
&+ (\gamma^1\gamma^2 + \gamma^2\gamma^1)\, p_1 p_2 \\
&+ (\gamma^1\gamma^3 + \gamma^3\gamma^1)\, p_1 p_3 \\
&+ (\gamma^2\gamma^3 + \gamma^3\gamma^2)\, p_2 p_3
\end{aligned}\right\}$$

<u>How</u> can we get rid of all "crossed" terms ?

$\longrightarrow$ certainly <u>not</u> with ordinary numbers !

Dirac's idea of choosing the $\gamma$'s as matrices:

$$(\gamma^0)^2 = 1 \qquad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \qquad \text{and}$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 0 \qquad \mu \neq \nu$$

or:

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$$

∘ recall:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

choosing $\gamma^0$ and $\gamma^i$ $(i = 1, 2, 3)$: 4 x 4 matrices

Bjorken-Drell convention

$$\boxed{\gamma^0 = \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} \hat{0} & \hat{\sigma}^i \\ -\hat{\sigma}^i & \hat{0} \end{pmatrix}}$$

Pauli matrices

We can then write:

$$(p^\mu p_\mu - m^2 c^2) = (\gamma^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc) = 0$$

choose one term: $\gamma^\mu p_\mu - mc = 0 \longrightarrow$

∘ Make the substitution : $p_\mu \rightarrow i\hbar\,\partial_\mu$

$$\boxed{i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0}$$

(Dirac equation)

$\psi$ is now a four-element column matrix:

$$\boxed{\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}}$$

$\psi_1, \psi_2$: spinor

$\psi_3, \psi_4$: spinor

note: This is not a vector (4-vector)!

Name: bi-spinor or Dirac spinor

## 4. Solutions to Dirac equation:

### a.) No position dependence:

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial z} = 0$$

(zero momentum)

In this case:

$$\frac{i\hbar}{c}\gamma^0 \frac{\partial \psi}{\partial t} - mc\,\psi = 0$$

$$\begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & -\hat{1} \end{pmatrix} \begin{pmatrix} \partial \psi_A / \partial t \\ \partial \psi_B / \partial t \end{pmatrix} = -i\,\frac{mc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

Where: $\psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ $\quad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$

This yields:

$$\frac{\partial \psi_A}{\partial t} = -i\left(\frac{mc^2}{\hbar}\right)\psi_A \qquad -\frac{\partial \psi_B}{\partial t} = -i\left(\frac{mc^2}{\hbar}\right)\psi_B$$

solutions:

$e^{-iEt/\hbar}$ $\quad e^{+iEt/\hbar}$

$$\psi_A(t) = e^{-i(mc^2/\hbar)t}\,\psi_A(0) \qquad \psi_B(t) = e^{+i(mc^2/\hbar)t}\,\psi_B(0)$$

positive energy solution: particle

negative energy solution: anti-particle

ignoring for the moment normalization factors we get 4 independent solutions to the Dirac equation:

$$\psi^{(1)} = e^{-i(mc^2/\hbar)t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \psi^{(2)} = e^{-i(mc^2/\hbar)t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

electron spin up — electron spin down

$$\psi^{(3)} = e^{+i(mc^2/\hbar)t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \psi^{(4)} = e^{+i(mc^2/\hbar)t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

positron spin up — positron spin down

Look now for plane wave solutions of the form:

$$\boxed{\psi(\vec{r}, t) = a\, e^{-i/\hbar \underbrace{(Et - \vec{p}\cdot\vec{r})}_{x^\mu p_\mu}} u(E, \vec{p})}$$

find a bi-spinor $u$ such that $\psi$ satisfies the Dirac equation

1. $$\partial_\mu \psi = -\frac{i}{\hbar} p_\mu\, a \cdot e^{-(i/\hbar)\, x^\mu p_\mu} \cdot u$$

2. Insert this into Dirac equation:

$$i\hbar \gamma^\mu \partial_\mu \psi - mc\psi = 0$$

$$i\hbar \gamma^\mu \left(-\frac{i}{\hbar}\right) p_\mu\, a\, e^{(-i/\hbar)\, x^\mu p_\mu} \cdot u - mc \cdot a \cdot e^{-(i/\hbar)\, x^\mu p_\mu} u = 0$$

Therefore:

$$\boxed{(\gamma^\mu p_\mu - mc)\, u = 0}$$

momentum space Dirac equation

- More explicitly:

$$\gamma^\mu p_\mu = \gamma^0 p^0 - \vec{\gamma}\cdot\vec{p} = \frac{E}{c}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \vec{p}\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} E/c & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E/c \end{pmatrix}$$

Therefore:

$$(\gamma^\mu p_\mu - mc)\underbrace{u}_{\begin{pmatrix} u_A \\ u_B \end{pmatrix}} = \begin{pmatrix} \left(\frac{E}{c} - mc\right)u_A - \vec{p}\cdot\vec{\sigma}\cdot u_B \\ \vec{p}\cdot\vec{\sigma}\cdot u_A - \left(\frac{E}{c} + mc\right)u_B \end{pmatrix} \stackrel{!}{=} 0$$

- Therefore:

$$\boxed{\begin{aligned} u_A &= \frac{\vec{p}\cdot\vec{\sigma}}{(E - mc^2)}\cdot u_B \\ u_B &= \frac{c}{E + mc^2}(\vec{p}\cdot\vec{\sigma})\, u_A \end{aligned}}$$

$$u_A = \frac{c^2}{E^2 - m^2c^4}(\vec{p}\cdot\vec{\sigma})^2 u_A$$

1.

$$(\vec{p}\cdot\vec{\sigma}) = p_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} p_z & (p_x - i p_y) \\ (p_x + i p_y) & -p_z \end{pmatrix}$$

2. $(\vec{p}\cdot\vec{\sigma})^2 = 1 \cdot p^2$ (check for your self)

Then:

$$u_A = \frac{c^2 \overbrace{\vec{p}^{\,2}}^{E^2 - m^2c^4}}{E^2 - m^2c^4}\, u_A \iff u_A = u_A$$

In order to satisfy the Dirac equation, E and p must satisfy the energy-momentum relation!

- Solution:

$$E = \pm \sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$$

positive: particle

negative: anti-particle

° Solutions : ( page 18)

| | |
|---|---|
| $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ | $u_B = \frac{c}{E + mc^2} (\vec{p} \cdot \vec{\sigma}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
| $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $u_B = \frac{c}{E + mc^2} (\vec{p} \cdot \vec{\sigma}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ |
| $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ | $u_A = \frac{c}{E - mc^2} (\vec{p} \cdot \vec{\sigma}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
| $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $u_A = \frac{c}{E - mc^2} (\vec{p} \cdot \vec{\sigma}) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ |

normalization : $u^+ u = 2|E| / c$

$$u = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \qquad u^+ = (\alpha^* \ \beta^* \ \gamma^* \ \delta^*)$$

transpose conjugate or

dagger Hermitian conjugate

Then : $u^+ u = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2$

Four solutions:

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{c p_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix} \qquad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{c(-p_z)}{E+mc^2} \end{pmatrix}$$

With: $E = +\sqrt{m^2c^4+\vec{p}^{\,2}c^2}$

$$u^{(3)} = N \begin{pmatrix} \frac{c(p_z)}{E-mc^2} \\ \frac{c(p_x+ip_y)}{E-mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad u^{(4)} = N \begin{pmatrix} \frac{c(p_x-ip_y)}{E-mc^2} \\ \frac{c(-p_z)}{E-mc^2} \\ 0 \\ 1 \end{pmatrix}$$

With: $E = -\sqrt{m^2c^4+\vec{p}^{\,2}c^2}$

Normalization: $N = \sqrt{(|E|+mc^2)/c}$

(Griffiths problem 7.3)

Introduce a <u>convention</u> by changing the sign for $E$ and $\vec{p}$ for the negative energy solution:

$$\boxed{\psi(\vec{r}, t) = a\, e^{i/\hbar\,(Et - \vec{p}\cdot\vec{r})}\, u(-E, -\vec{p})}$$

<u>(for solution 3 and 4)</u>

- <u>Use new symbol $v$ for positrons (anti-particles):</u>

$$v^{(1)}(E, \vec{p}) = u^{(4)}(-E, -\vec{p}) = N \begin{pmatrix} \frac{c(p_x - i p_y)}{E + mc^2} \\ \frac{c(-p_z)}{E + mc^2} \\ 0 \\ 1 \end{pmatrix}$$

$$v^{(2)}(E, \vec{p}) = -u^{(3)}(-E, -\vec{p}) = N \begin{pmatrix} \frac{c(p_z)}{E + mc^2} \\ \frac{c(p_x + i p_y)}{E + mc^2} \\ 1 \\ 0 \end{pmatrix}$$

- <u>Finally:</u>

Particles $\quad \boxed{(\gamma^\mu p_\mu - mc)\, u = 0}$

Anti-particles $\quad \boxed{(\gamma^\mu p_\mu + mc)\, v = 0}$ (different sign for $p_\mu$)

## 5. Bilinear covariants:

The components of a Dirac spinor do not transform as a four vector:

See Bjorken & Drell, QFT I chapter 2

$$\psi \to \psi' = S\psi \qquad S = a_+ + a_- \gamma^0 \gamma^1 = \begin{pmatrix} a_+ & a_- \sigma_1 \\ a_- \sigma_1 & a_+ \end{pmatrix}$$

$$a_\pm = \pm \sqrt{\tfrac{1}{2}(\gamma \pm 1)}$$

transformation to a system moving with speed $v$ in " direction

$$\gamma = 1 / \sqrt{1 - v^2/c^2}$$

adjoint spinor

- Scalar quantity: $\overline{\psi} = \psi^+ \gamma^0 = (\psi_1^* + \psi_2^* - \psi_3^* - \psi_4^*)$

$\overline{\psi}\,\psi$

- Then:

$$\overline{\psi}\,\psi = \psi^+ \gamma^0 \psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^3 - |\psi_4|^2$$

This quantity is relativistic invariant:

$$(\overline{\psi}\psi)' = (\psi')^+ \gamma^0 \psi' = \psi^+ \underbrace{S^+ \gamma^0 S}_{\gamma^0} \psi = \psi^+ \gamma^0 \psi = \overline{\psi}\psi$$

Problem 7.11

There 16 products of the form $\psi_i^* \psi_j$

$(i, j = 1, \ldots, 4)$ :  $\gamma^5 = i\gamma^0\gamma^1\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

1. $\bar{\psi}\psi$ : scalar : one component
2. $\bar{\psi}\gamma^5\psi$ : pseudo scalar : one component
3. $\bar{\psi}\gamma^\mu\psi$ : vector : four components
4. $\bar{\psi}\gamma^\mu\gamma^5\psi$ : pseudo vector : four components
5. $\bar{\psi}\sigma^{\mu\nu}\psi$ : anti sym. tensor : six componenents

Where :

$$\boxed{\sigma^{\mu\nu} = \frac{i}{2i}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)}$$