

Lecture 5: Symmetries and Invariance
Principles - Part I

Overview:

1. The Noether Theorem
2. Formal aspects of group theory
3. Lie groups and Lie algebras
4. $SU(2)$
5. Isospin

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1. $SU(3)$
 2. Discrete symmetries: P, C, G
 3. CP - K-system
 4. CPT

1. The Noether Theorem :

The Noether theorem plays a central role in theoretical physics. It allows to relate basic ideas of physics:

- a) invariance of the form that a physical law stays with respect to any (generalized) transformation and
- b) conservation law of a physical quantity.

Noether's theorem (Emmy Noether, 1917) :

To every symmetry, there is a corresponding conservation law and vice versa!

Examples:

1. Invariance of a physical system under translation:
→ Conservation of momentum
2. Invariance of a physical system under rotation:
→ Conservation of angular momentum
3. Invariance of a physical system under time:
→ Conservation of energy

The Noether theorem:

Define a set of transformations:

time : $t' = t'(t)$

space : $q_i' = q_i'(q_1, \dots, q_f, t)$

(1)

For the inverse operation:

$$t = t(t'); \quad q_i = q_i(q_1', \dots, q_f', t') \quad (2)$$

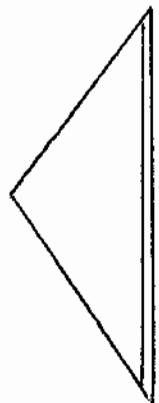
A system is described by a given Lagrange function $L(q_1, \dots, \dot{q}_1, \dots, t)$. With the above transformations in (2), we get:

$$\underline{L(q_1, \dots, \dot{q}_1, \dots, t) dt = L'(q_1', \dots, \dot{q}_1', \dots, t') dt' \quad (3)}$$

new Lagrangian L' depending on
 q_1', \dots, t'

Goal: Find the condition in which the equations of motion have the same form as in the old variables.

Such a transformation exists if the new Lagrange function $L'(q_1', \dots, \dot{q}_1', \dots, t')$ equals to the old Lagrange function $L(q_1, \dots, \dot{q}_1, \dots, t)$ or differs by the total differential of a function $\varphi(q_1, \dots, \dot{q}_1, \dots, t')$.



$$L(\dot{q}_1, \dots, \dot{q}_i, \dots, t) = L(q_1, \dots, q_i, \dots, t) + \frac{d\mathcal{L}}{dt}(q_1, \dots, t) \quad (4)$$

together with equation (3) we find:

$$\int L(q_1, \dots, \dot{q}_i, \dots, t) dt = \int L(q_1, \dots, \dot{q}_i, \dots, t) dt + d\mathcal{L}(q_1, \dots, t) \quad (5)$$

Let us define a set of transformations. Provided that the symmetry transformations satisfies a continuous group, it is sufficient to consider only infinitesimal transformations (we will come back to this later):

$$\begin{aligned} t' &= t + \delta t \\ q'_i &= q_i + \delta q_i \\ \dot{q}'_i &= \dot{q}_i + \delta \dot{q}_i \end{aligned} \quad (6)$$

before we consider a transformation of that type on equation (5), let us derive a few important relations.

$$\text{a) } \dot{\delta q_i} = \frac{d}{dt} \delta q_i - \dot{q}_i \frac{d}{dt} st :$$

$$\dot{q}_i = \frac{d}{dt} \dot{q}_i = \frac{d}{dt} \dot{q}_i \frac{dt}{dt} = \frac{d}{dt} (\dot{q}_i + \dot{\delta q}_i) \frac{dt}{dt}$$

Now we:

$$\frac{dt}{dt} = \frac{1}{\frac{dt}{dt}} = \frac{1}{\frac{dt}{dt} + \frac{d}{dt} st} = \frac{1}{1 + \frac{d}{dt} st}$$

With: $\frac{1}{(1+x)} \approx 1-x$ for $x \ll 1$ we find

$$\begin{aligned} \dot{\delta q}_i &= \dot{q}_i - \dot{q}_i = \frac{d}{dt} (\dot{q}_i + \dot{\delta q}_i) \frac{dt}{dt} - \dot{q}_i = \\ &= (\dot{q}_i + \frac{d}{dt} \dot{\delta q}_i) \left(1 - \frac{d st}{dt} \right) - \dot{q}_i = \end{aligned}$$

$$\begin{aligned} \dot{\delta q}_i &= \dot{q}_i - \dot{q}_i \frac{d st}{dt} + \frac{d \dot{\delta q}_i}{dt} - \underbrace{\frac{d \dot{\delta q}_i}{dt} \cdot \frac{d st}{dt}}_{O(\delta^2)} - \dot{q}_i = \\ &= \frac{d}{dt} \dot{\delta q}_i - \dot{q}_i \frac{d st}{dt} \end{aligned}$$

therefore:

$$\boxed{\dot{\delta q}_i = \frac{d}{dt} \dot{\delta q}_i - \dot{q}_i \frac{d}{dt} st} \quad (7)$$

b) $\delta(L dt)$:

$$\delta(L dt) = \delta L dt + L dt =$$

$$= \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \cdot \delta \dot{q}_i \right) dt + \left(\frac{\partial L}{\partial t} \right) \delta t dt + L d\delta t$$

with $\delta \dot{q}_i = \frac{d}{dt} \delta q_i - \dot{q}_i \frac{d}{dt} \delta t$ we get:

$$\delta(L dt) = \sum_i \left(\frac{\partial L}{\partial q_i} \cdot \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \cdot \frac{d}{dt} \delta q_i \right) dt + \left(\frac{\partial L}{\partial t} \right) \delta t dt +$$

$$(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \cdot \dot{q}_i) \frac{d\delta t}{dt} dt$$

With these two relations (equation 7 and 8) we can now consider the infinitesimal transformations in equation (5):

$$L(q_1, \dots, \dot{q}_1, \dots, t) dt = L(\underline{q_1 + \delta q_1}, \dots, \underline{\dot{q}_1 + \delta \dot{q}_1}, \dots, \underline{t + \delta t}) d(\underline{t + \delta t}) \\ + d\delta U(q_1, \dots, t)$$

We can rewrite this as:

$$\int L(q_1, \dots, \dot{q}_1, \dots, t) dt = L(q_1 + \delta q_1, \dots, \dot{q}_1 + \delta \dot{q}_1, \dots, t + \delta t) dt \\ + L \cdot \delta t + d \delta U$$

$$\delta L dt + L dt + d \delta U = 0$$

Therefore:

$$\delta(L dt) + d \delta U = 0 \quad (9)$$

Putting now equation (8) into (9):

$$\sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta \dot{q}_i \right) + \left(\frac{\partial L}{\partial t} \right) \cdot \delta t \\ + \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \frac{d \delta t}{dt} = - \frac{d \delta U}{dt} \quad (10)$$

Use now the following equations to formulate an equation as: $\frac{d}{dt} [\dots] = 0$

$$\frac{d}{dt} [\dots] \xrightarrow{\text{const.}} \underline{\text{const.}}$$

$$1. \quad \frac{d}{dt} \frac{\frac{\partial L}{\partial \dot{q}_i}}{\frac{\partial L}{\partial \dot{q}_i}} = \frac{\frac{\partial L}{\partial \dot{q}_i}}{\frac{\partial L}{\partial \dot{q}_i}}$$

$$2. \quad \frac{d}{dt} \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) = \left(\frac{\partial L}{\partial t} \right)$$

now now:

$$\frac{d}{dt} \left\{ \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) \delta t \right\} =$$

$$\frac{d}{dt} \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \cdot \dot{q}_i \right) \delta t + \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \cdot \dot{q}_i \right) \frac{d \delta t}{dt}$$

$\frac{\partial L}{\partial t} \cdots \delta t$

3. with: $\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right)$ we find:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) = \frac{\partial L}{\partial \dot{q}_i} \cdot \delta \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \cdot \frac{d}{dt} \cdot \delta \dot{q}_i$$

this provides the following relation for $\frac{d}{dt} [\dots] = 0$

$$\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \cdot \delta \dot{q}_i + \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \cdot \dot{q}_i \right) \delta t + \delta L \right] = 0$$

That means:

$$\boxed{\sum_i \left(\frac{\partial L}{\partial \dot{q}_i} \right) \cdot \delta \dot{q}_i + \left(L - \sum_i \frac{\partial L}{\partial \dot{q}_i} \cdot \dot{q}_i \right) \delta t + \delta L = \text{const.}}$$

Noether Theorem

(11)

Let's go back to our examples:

1. Translation: $\delta x_3 = \text{const.}; \delta x_1 = \delta x_2 = 0; \delta t = 0$
 $\delta L = 0$

This gives with equation (11):

$$\frac{\partial L}{\partial \dot{x}_3} = p_3 = \text{const.}$$

conservation of
momentum

2. Rotation: $\delta p_3 = \text{const.}; \delta \varphi_1 = \delta \varphi_2 = 0; \delta t = 0; \delta L = 0$

$$\frac{\partial L}{\partial \dot{\varphi}_3} = l_3 = \text{const.}$$

conservation of
angular momentum

3. Time: $\delta t = \text{const.}; \delta q_i = 0; \delta L = 0$

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \cdot \ddot{q}_i - L = E = \text{const.}$$

conservation of
energy

2. Formal aspects of group theory:

Group theory is the branch of mathematics that underlies the treatment of symmetry.

Example: rotation groups

Given are a set of rotations R_1, R_2 and R_3 . The set of rotations form a group. Each rotation is an element of the group.

Definition: A group is a set G in which a multiplication operation \cdot is defined with the following properties:

1. If R_i and R_j are in G , $R_i \cdot R_j$ is in G (Closure)
2. There is an identity element I in G such that $I \cdot R_i = R_i \cdot I = R_i$ for any R_i in G (Identity)
3. For every R_i in G , there is an inverse element in G called R_i^{-1} such that: $R_i \cdot R_i^{-1} = R_i^{-1} \cdot R_i = I$ (Inverse)
4. For every R_i, R_j, R_k in G :
 $(R_i \cdot R_j) \cdot R_k = R_i \cdot (R_j \cdot R_k)$ (Associativity)

• Note : A group is called non-Abelian if the following holds : $R_i \cdot R_j \neq R_j \cdot R_i$ and vice versa.

1. Transformations in space and time form an Abelian group.
2. Rotations form a non-Abelian group

• Groups can be :

- a) finite or infinite
- b) continuous or discrete

The rotation group is a continuous group in that each rotation can be labeled by a set of continuously varying parameters $(\alpha_1, \alpha_2, \alpha_3)$.

The rotation group is a Lie group. The rotation can be expressed as the product of a succession of infinitesimal rotations \rightarrow the group is completely defined by the 'neighborhood of the identity'.

In quantum mechanics, a transformation of the system is associated with a unitary operator U in Hilbert space :

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle$$

Review on Matrix algebra:

1. Unitary matrix:

A square matrix U is a Unitary matrix if $U^* = U^{-1}$ where U^* denotes the adjoint matrix and U^{-1} is the inverse matrix.

2. Adjoint matrix:

The adjoint matrix, sometimes called the adjugate matrix, Hermitian transpose, is defined by $U^* = \bar{U}^T$

Here U^T denotes the transpose of matrix U (replace matrix elements u_{ij} by u_{ji}) and \bar{U} denotes the conjugate matrix (replace matrix u_{ij} by the complex conjugate \bar{u}_{ij}).

3. Hermitian matrix:

A square matrix H is called Hermitian if it is self-adjoint:

$$H = H^*$$

Example: Pauli matrices

4. orthogonal matrix:

A $n \times n$ matrix O is an orthogonal matrix, if
 $A \cdot A^T = 1$, i.e. $A^{-1} = A^T$ with: $(A^{-1})_{ij} = a_{ji}$

Note:

1. A unitary matrix U is called special unitary matrix, if:

$$U U^* = 1 \quad \text{and} \quad \det U = 1 \quad SU$$

2. A orthogonal matrix O is called special orthogonal matrix, if:

$$O \cdot O^T = 1 \quad \text{and} \quad \det O = 1 \quad SO$$

Important groups in elementary particle physics:

Group name:

matrices in group:

$U(n)$

$n \times n$ Unitary

$\rightarrow SU(n)$

$n \times n$ Unitary w/ determ. 1

$O(n)$

$n \times n$ Orthogonal

$SO(n)$

$n \times n$ Orthogonal w/ determ. 1

A transformation group of a quantum mechanical system is associated with a mapping of the group into a set of unitary operators.

For each x in G , there is a $U(x)$ which is a unitary operator: $x \rightarrow U(x)$

group operations are preserved: $U(x) \cdot U(y) = U(xy)$

such a mapping is called representation.

Example: $U(n) = e^{in\theta}$ is a representation of the additive group of integers:

$$e^{in\theta} \cdot e^{im\theta} = e^{i(n+m)\theta}$$

Note: 1. It is convenient to view representations as abstract linear operators and as matrices.

2. Two representations U_1 and U_2 are equivalent if they are related by a similarity transformation:

$$U_2 = S U_1 S^{-1}$$

3. A representation U is reducible if it is equivalent to a representation U' with block-diagonal form:

$$U = S U' S^{-1} = \begin{vmatrix} U'_1(x) & 0 \\ 0 & U'_2(x) \end{vmatrix}$$

4. The representation U' is said to be the direct sum of U'_1 and U'_2 :

$$U' = U'_1 \oplus U'_2$$

5. A representation is irreducible if it is not reducible, that is if it cannot be put into block diagonal form by a similarity transformation.

→ we will almost never talk about the group elements as abstract mathematical objects, but

in terms of their representations: operators ("matrix")

3. lie-groups and lie algebras:

Compact lie groups are groups of unitary operators in which the group elements are labeled by a set of continuous parameters.

Any unitary matrix can be written as:

$$U = e^{iH} = 1 + iH - \frac{1}{2!} \cdot H^2 + \dots$$

H : Hermitian, trace less matrix

In a lie group, the elements of the group are characterized by a finite number of real parameters a_α . For SU(n) one has $n^2 - 1$ real parameters, the number of independent parameters for an arbitrary, trace less, Hermitian matrix.

• Note:

DO NOT mix up dimension of L_α from dimension $n^2 - 1$!

e.g.: SU(2)

Parameters

generators $N = n^2 - 1 = 3$

→ 3 generators!

- Again: For $SU(2)$: 3 generators
The dimension of those generators depends on the quantum mechanical system under consideration:
 - spin $\frac{1}{2}$ particles
 - spin 1 particles

In general:

To study the representations, it is sufficient to study the generators :

$$[L_\alpha, L_\beta] = i \epsilon_{\alpha\beta\gamma} L_\gamma$$

The generators and their commutation relations specify a Lie algebra where the $\epsilon_{\alpha\beta\gamma}$ are called the structure constants.

Jacobi identity:

$$[L_\alpha, [L_\beta, L_\gamma]] + \text{cycl. perm.} = 0$$

→ Simplest non-Abelian Lie algebra

$$N=3, SU(2) : \quad \epsilon_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}$$

anti-sym. tensor

4. SU(2) :

1. General :

We first outline the construction of representations of $SU(2)$ in a systematic way. We want to construct the Hermitian representation matrices $\vec{S} = (S_1, S_2, S_3)$ that satisfy:

$$[S_i, S_j] = i \epsilon_{ijk} S_k$$

Casimir operator:

Except for operators from the set of generators there are other operators that can be constructed from them and commute with all generators of the group so called Casimir operators:

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

Extract construction from the commuting rules:

$$S_{\pm} = S_x \pm i S_y$$

$$[S_z, S_{\pm}] = \pm S_{\pm}$$

$$[S_+, S_-] = 2S_z$$

$$[S_x^2, S_z] = 0$$

$$S^2 |S, m\rangle = s(s+1) |S, m\rangle$$

$$S_z |S, m\rangle = m |S, m\rangle$$

spin s

Note: $m = -s, -s+1, \dots, s$
 s can take on any value $0, \frac{1}{2}, 1, \dots$

Matrix representation:

a) Singlet: 1-dimensional representation : spin 0

$$|0, 0\rangle$$

$$S_z = (0); \quad S_+ = (0); \quad S_- = (0)$$

b) 2 dimensional repr. : spin $\frac{1}{2}$

For $SU(2)$, the 2-dim. repr. has the basis states

$$|\sigma = \frac{1}{2}, m = \frac{1}{2}\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

"spin up"

$$|\sigma = \frac{1}{2}, m = -\frac{1}{2}\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

"spin down"

• representation matrices:

$$\sim S_x = \begin{pmatrix} 0 & 1/2 \\ 0 & -1/2 \end{pmatrix} \quad S_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad S_z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

generators of $SU(2)$ for spin $1/2$

c) 3 dim. repr. : spin 1

$$|1, -1\rangle \quad |1, 0\rangle ; \quad |1, 1\rangle$$

matrix representation:

$$\sim S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad S_y = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad S_z = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

generators of $SU(2)$ for spin 1

• Combining representations:

$$\hat{S} = \hat{S}_A + \hat{S}_B \longrightarrow | s_A, m_A \rangle ; | s_B, m_B \rangle$$

$$S = | s_A - s_B |, | s_A - s_B | + 1, \dots, s_A + s_B$$

$$M = m_A + m_B$$

$$| s_A s_B, S M \rangle = \sum_{m_A, m_B} c(m_A m_B; S M) | s_A s_B m_A m_B \rangle$$

m_A, m_B

Clebsch-Gordan coefficient

• Other useful notation for combined representation:

a.)

$$2 \otimes 2 = 3 \oplus 1 \quad \text{or} \quad \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

2 spin $\frac{1}{2}$ system : 2-dim-repr.

b.) for 3 spin $\frac{1}{2}$ particles:

irred.
repr.

$$2 \otimes 2 \otimes 2 = (3 \otimes 2) \oplus (1 \otimes 2) = 4 \otimes 2 \otimes 2$$

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (1 \otimes \frac{1}{2}) \oplus (0 \otimes \frac{1}{2}) = \frac{3}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$$

5. ISO spin: $\rightarrow \text{SU}(2)$

Hilbert observed shortly after the discovery of the neutron in 1932 that the neutron is almost equal to the proton apart from their respective charge.

$$m_p = 938.28 \text{ MeV}/c^2 \quad m_n = 939.57 \text{ MeV}/c^2$$

Hilbert proposed that one regards neutrons and protons as "two states" of a single particle, the nucleon:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

ISO spin I With 3 generators: I_1, I_2, I_3

$$p = \left| \frac{1}{2} \frac{1}{2} \right\rangle \quad n = \left| \frac{1}{2} -\frac{1}{2} \right\rangle$$

NOW: Strong force is invariant under rotations in ISO spin space

\rightarrow Noether theorem: ISO spin is conserved?

1.

nucleon: two dimensional repr. $SU(2)$ isospin $\frac{1}{2}$
 \sim $I = 1/2$

2.

pions: $I = 1$ 3-dim. repr.

$$\bar{\pi}^+ = |11\rangle ; |10\rangle ; \bar{\pi}^- = |1-1\rangle$$

3. Δ , $I = 0$: 1-dim. repr.

$$\Lambda = |0, 0\rangle$$

4 Δ' 's, $I = 3/2$: 4-dim. repr.

$$\Delta^{++} = \left| \frac{3}{2}, \frac{3}{2} \right\rangle ; \Delta^+ = \left| \frac{3}{2}, \frac{1}{2} \right\rangle ; \Delta^0 = \left| \frac{3}{2}, -\frac{1}{2} \right\rangle ; \Delta^- = \left| \frac{3}{2}, -\frac{3}{2} \right\rangle$$

general: multiplicity: $2I + 1$

generators: isospin operators

$$\hat{T}_i = \frac{1}{2} \tau_i \quad (i=1,2,3) \quad \tau_i: \text{Pauli matrix}$$

$$[\hat{T}_i, \hat{T}_j] = i \epsilon_{ijk} \hat{T}_k$$

Compound systems : Isospin

2-nucleon system

$$|11\rangle = |pp\rangle$$

$$|10\rangle = (1/\sqrt{2})(pn + np)$$

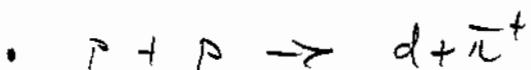
$$|1-1\rangle = nn$$

$$|00\rangle = (1/\sqrt{2})(pn - np) \quad \text{ deuteron : isosinglet}$$

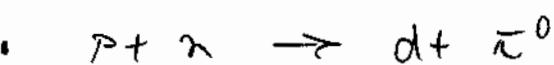
Symmetrical importance of isospin in variance:

deuteron : $I=0$

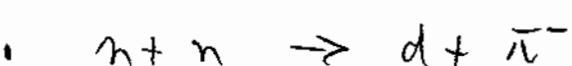
nucleon-nucleon scattering



$$|11\rangle \quad |11\rangle$$



$$|10\rangle \quad (1/\sqrt{2})(|10\rangle + |00\rangle)$$



$$|1-1\rangle \quad |1-1\rangle$$

$$M_a : M_b : M_c = 1 : (1/2) : 1$$

since: cross-section $\propto |M|^2$

$$\rightarrow \sigma_a : \sigma_b : \sigma_c = 2 : 1 : 2$$

only $I=1$
contribute