Lecture 9: Introduction to practical aspects of quantum field theory:

Overview:

1. Lifetime and cross-sections - general comments
2. Illustrative example of the concept of cross-section (Hard sphere scattering)
3. S Matrix formalism - general concepts
4. ~~Fermi's golden rule~~
5. Lagrange formalism and field quantization concepts
   - (a.) classical case
   - (b.) quantization
   - (c.) Feynman rules
6. Examples for phase space integrations for various processes
   - (a.) Two-body decay
   - (b.) Two-body scattering

1. Life time and cross-sections - general comments

Three experimental probes of elementary particle interactions:

- bound states : } non-relativistic QM (e.g. heavy quarks)
- decays : } relativistic quantum field theory
- scattering

  - single-particle theory
  - many-particle theory: number of particles of a given type is not constant: annihilation and creation of particles!

This is reflected in the underlying quantum-theory:

- non-relativistic quantum mechanics: Schrödinger equation
- relativistic quantum mechanics: Klein-Gordon equation, Dirac equation

} "1st quantisation"

- Quantum field theory: Quantising fields: fields are operators

} "2nd quantisation"

- 1st quantisation : ordinary quantum mechanics

Heisenberg commutation relations:

$$\boxed{\begin{array}{ll} [x_i, p_j] = i\,\delta_{ij} & (i,j = 1,2,3) \\ [x_i, x_j] = [p_i, p_j] = 0 & \end{array}}$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \psi + V(\vec{x},t)\,\psi$$

x_i, p_i : operators solution: wave function $\psi(x,t)$

- 2nd quantisation:

"wave functions" are treated as quantised fields which are reflected by operators.

Those fields are subject to a set of commutation rules ("canonical quantisation"):

~ bosons : commutation relations

~ fermions : anti-commutation relations

The fields under consideration can be expressed as Fourier expansions with the annihilation and creation operators as coefficients.

The above commutation relations for fields translate then into commutation (anti-commutation) relations for annihilation and creation operators!

Those operators provide then a natural interpretation for the annihilation and creation of particles in

high - energy decay

high - energy scattering processes.

A second approach is to formulate the quantisation of fields through method of path integrals. This method can be employed not only in quantum field theory, but also in ordinary quantum mechanics. :

The path integral formulation of quantum mechanics is based directly on the notation of a propagator $K(q_f t_f; q_i t_i)$.

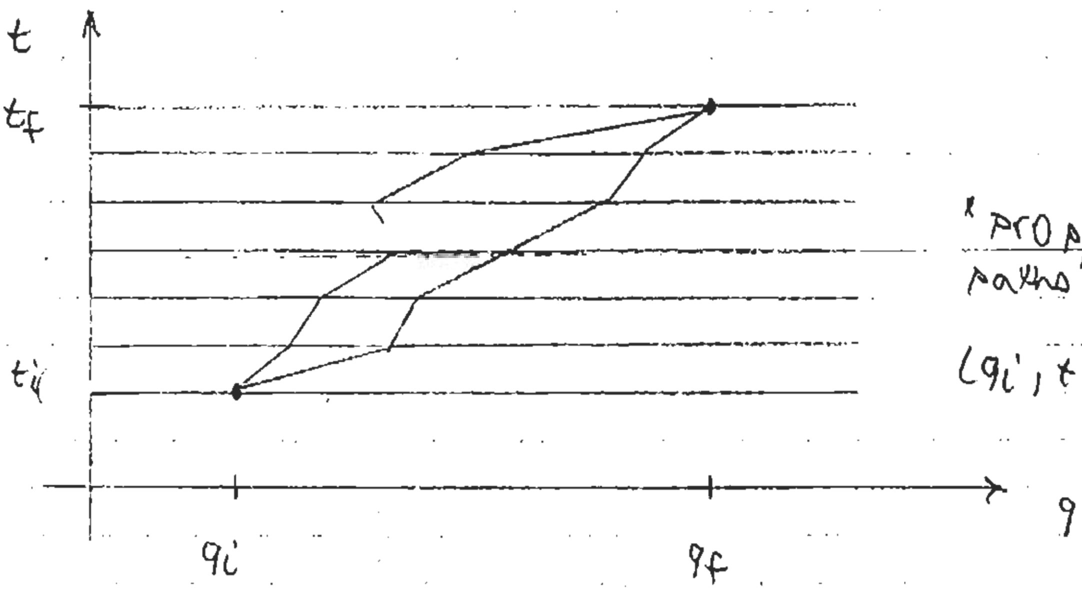
Given a wave function $\psi(q_i, t_i)$ at time t_i, the propagator gives the corresponding wave function at a later time t_f :

$$\boxed{\psi(q_f, t_f) = \int K(q_f t_f; q_i t_i)\, \psi(q_i t_i)\, dq_i}$$

The propagator is nothing else then:

$$\boxed{K(q_f t_f ; q_i t_i) = \langle q_f t_f | q_i t_i \rangle}$$

Aim: Path integral formulation of $\langle q_f t_f | q_i t_i \rangle$:



"propagation over many paths" from

$(q_i, t_i) \longrightarrow (q_f, t_f)$

Consider all possible paths:

classical Lagrange function

$$\boxed{\langle q_f t_f | q_i t_i \rangle = N \int \mathcal{D}q \, \exp\left[\frac{i}{\hbar} \underbrace{\int_{t_i}^{t_f} L(q, \dot{q})\, dt}_{S:\ \text{action}}\right]}$$

Infinite differential of a function

Each function $q(t)$ and $p(t)$ defines a path in phase space!

Note:

1. We have an explicit expression within this Path integral formulation for the transition amplitude which is well suited for application in scattering problems.

2. The "quantisation" comes into the game that we are considering all possible paths for the transition from the initial to final state and that we are summing over all possible paths.

3. Last important comment:

   a.) The path integral formalism and the method of canonical quantisation lead to the same result. Both formulations are equivalent!

Quantisation of fields with boundary conditions (e.g.: Gauge theories: QED and QCD) is rather difficult for canonical quantisation.

Path integral method: preferred approach!

4. Most important approach of calculating transition amplitude:

series expansion in coupling constant for a given interaction.

The individual terms in each order can be "graphically interpreted" which we already know. Those are Feynman graphs.

A set of rules exist which can be derived for a given theory, quantified by the respective Lagrangian, to translate a 'Feynman graph' into a mathematical expression to calculate a transition amplitude and therefore provide the theoretical basis for high-energy decays and scattering processes.

- Note: • 'Feynman rules' can be derived from the canonical quantisation approach as well as through the method of path integrals.

  • 'Feynman rules' provide a 'tool-box' like approach to high-energy processes.

  $\longrightarrow$ The following discussion will be restricted to apply these rules!

The commonest types of process which concern the elementary particle physicist are:

a.) scattering processes

⟶ measure cross-section for a particular reaction

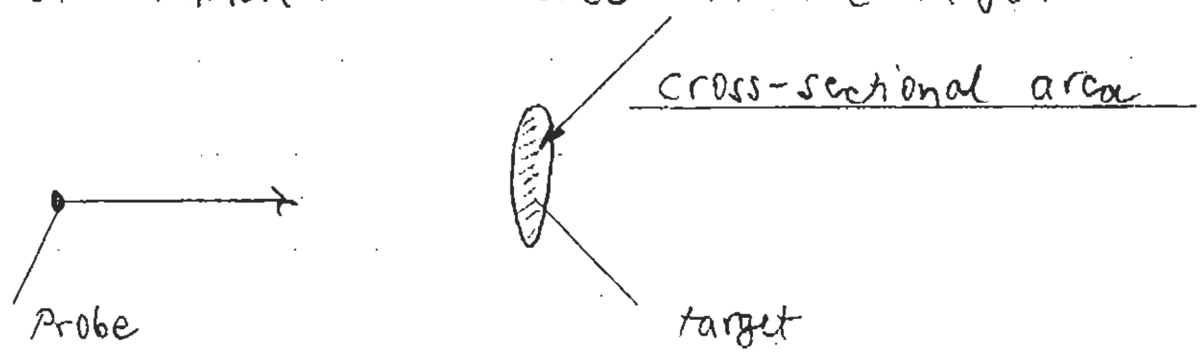
b.) decay of one particle

⟶ measure decay width / lifetime

Before we start with the calculation of these quantities let us first provide a clear definition and understanding:

a.) cross-section:

Parameter of interest: "size of the target"



In the microscopic world:

1. No sharp edges; the probe particle is more or less deflected

2. "cross-section" depends on nature of probe (compare electron vs. neutrino) as well as the target

3. It depends on the out going particle :

a) elastic scattering : $e + p \rightarrow e + p$

At high enough energies : inelastic scattering

e.g. $e + p \rightarrow e + X$ with

$X = p + \gamma$ or $X = p + \pi^0$

Each individual process has its own cross-section

σ_i $\longrightarrow$ exclusive (e.g. only $p + \gamma$ for X)

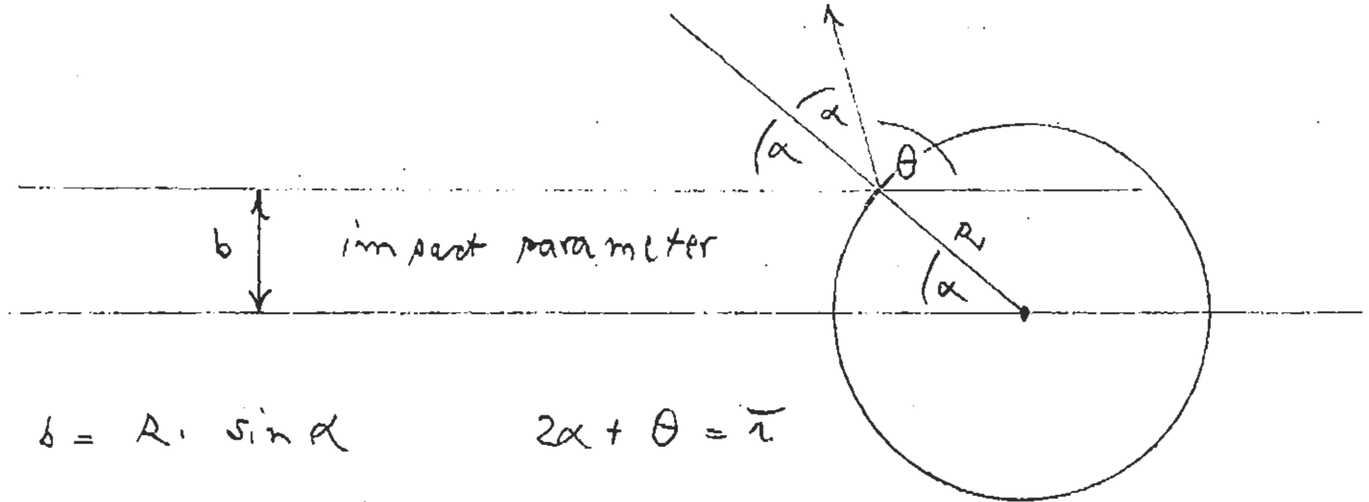
The total case refers to considering all final states

for X $\longrightarrow$ inclusive

$$\boxed{\sigma_{tot} = \sum_{i=1}^{n} \sigma_i}$$

Units : 1 barn $= 10^{-24}$ cm^2

2. illustrative example for the 'meaning' of the term cross-section: Hard-sphere scattering



$b = R \cdot \sin\alpha \qquad 2\alpha + \theta = \pi$

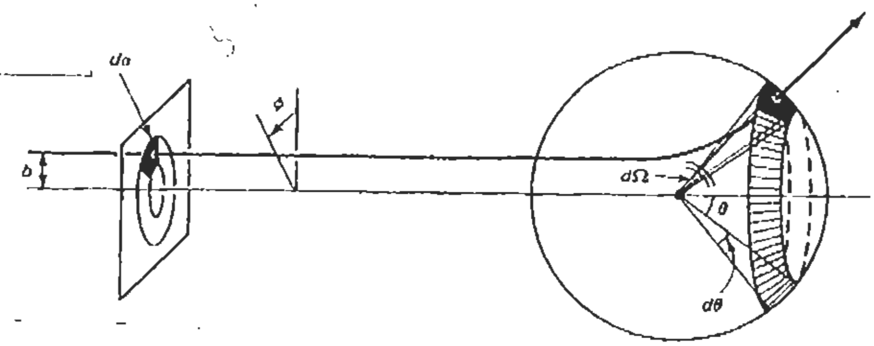
$\sin\alpha = \sin(\pi/2 - \theta/2) = \cos(\theta/2)$

$b = R \cdot \cos(\theta/2) \qquad \boxed{\theta = 2 \arccos(b/R)}$

Differential cross-section:

$d\sigma = |b\,db\,d\phi|$

$d\Omega = |\sin\theta\,d\theta\,d\phi|$



From above:

$$\boxed{\left(\frac{db}{d\theta}\right) = -\frac{R}{2} \cdot \sin\left(\frac{\theta}{2}\right)}$$

$$D(\theta) = \frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\theta} \cdot \left(\frac{db}{d\theta}\right) \right| = \frac{Rb\sin(\theta/2)}{2\sin\theta} = \frac{R^2}{2} \cdot \frac{\cos(\theta/2) \cdot \sin(\theta/2)}{\sin\theta} = \frac{R^2}{4}$$

- Total cross-section:

$$\sigma = \int d\sigma = \int D(\theta)\, d\Omega = \int \frac{R^2}{4}\, d\Omega = \underline{\pi R^2}$$

Total cross section the sphere presents to an incoming beam.

- Note: – Any particle within this area will scatter
  – Any particle outside this area will pass by unaffected!

- Rutherford scattering:

Relation between impact parameter and scattering angle (c.f. Goldstein):

$$\boxed{b = \frac{q_1 q_2}{2E} \cdot \cot(\theta/2)}$$

$$\boxed{\left(\frac{d\sigma}{d\theta}\right) = \left(\frac{q_1 q_2}{4E \sin^2(\theta/2)}\right)^2}$$

- Luminosity: number of particles per unit time, per unit area

$$dN = \mathcal{L}\, d\sigma = \mathcal{L}\, D(\theta) \cdot d\Omega$$

$$\boxed{\frac{d\sigma}{d\Omega} = D(\theta) = \frac{1}{\mathcal{L}} \cdot \frac{dN}{d\Omega}}$$

Number of particles per unit time scattered into solid angle $d\Omega$, divided by $d\Omega$ and by the luminosity.

$$\left(\frac{d\sigma}{d\Omega}\right)$$

Decay rate:

$$dN = -\Gamma N\,dt$$

$$\boxed{N(t) = N(0)\,e^{-\Gamma_i t}}$$

decay rate

lifetime: $$\boxed{\tau = \frac{1}{\Gamma_{tot}}}$$

Total decay rate: $$\Gamma_{tot} = \sum_{i=1}^{n} \Gamma_i$$

individual decay rat for a particular decay mode

Branching ratio: B_i

$$\boxed{B_i = \frac{\Gamma_i}{\Gamma_{tot}}}$$

3. S matrix formalism – general concepts

initial state: $|i\rangle$ $\qquad \langle i|i\rangle = 1$

final state: $|f\rangle$

The probability to find $|f\rangle$ is given by the square of the S matrix element:

$$|\langle f|S|i\rangle|^2$$

sum over all final states:

$$\sum_f |\langle f|S|i\rangle|^2 = \sum_f \langle i|S^+\underbrace{|f\rangle\langle f|}_{1}S|i\rangle = \langle i|S^+S|i\rangle = 1$$

S: unitary scattering matrix

more specific:

$$\boxed{a_1(p_1) + \dots + a_n(p_n) \to b_1(p_1') + \dots + b_m(p_m')}$$

- $\lim\limits_{t\to-\infty} |t\rangle = |i\rangle$
  $= |a_1(p_1) \dots a_n(p_n)\rangle$

- $\lim\limits_{t\to+\infty} |t\rangle = |f\rangle$
  $= |b_1(p_1') \dots b_m(p_m')\rangle$

S-Matrix: $\boxed{S_{fi} = \langle b_1(p_1') \dots b_m(p_m')|S|a_1(p_1) \dots a_n(p_n)\rangle}$

separate out the non-interacting part:

$$S_{fi} = \delta_{fi} + \text{rest}$$

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta(P_f - P_i) \langle f | T | i \rangle$$

(Definition of T or Transition matrix)

∘ where:

$P_f = p_1' + \dots + p_m'$

$P_i = p_1 + \dots + p_n$

$$\langle f | T | i \rangle \equiv \langle b_1(p_1') \dots b_m(p_m') | T | a_1(p_1) \dots a_n(p_n) \rangle = \mathcal{M}$$

∴ Fermi's golden rule:

2 fundamental ingredients:

- amplitude $\mathcal{M}$: ∘ dynamical aspects
- phase space : • kinematical information

(process is more likely to occur the larger the final state phase space)

e.g. decay of particle

However, dynamical aspects of the underlying theory might restrict this!

Example:

- Fermi's Golden rule:

The transition rate for a given process is determined by the amplitude and phase space according to Fermi's golden rule:

$$\boxed{\text{transition rate} = \frac{2\pi}{\hbar} |\mathcal{M}|^2 \times (\text{phase space})}$$

a.) Decays:

We generally consider the decay of a particle 1 to $(n-1)$ particles:

$$1 \rightarrow 2 + 3 + \dots + n$$

Decay rate:

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2\hbar m_1} \left[\left(\frac{c\, d^3\vec{p}_2}{(2\pi)^3 2E_2} \right) \cdot \left(\frac{c\, d^3\vec{p}_3}{(2\pi)^3 2E_3} \right) \cdots \left(\frac{c\, d^3\vec{p}_n}{(2\pi)^3 2E_n} \right) \right.$$

$$\times (2\pi)^4 \cdot \delta^4(p_1 - p_2 - p_3 - \dots - p_n)$$

Note: $p_i = (E_i/c, \vec{p}_i)$: four momentum of ith particle

$p_1 = (m_1 c, \vec{0})$: Decaying particle is at rest

S : statistical factor. $1/j!$ for each group j identical particles in the final state

- Total decay rate:

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{c}{4\pi}\right)^2 \frac{1}{2}\int \frac{|\mathcal{M}|^2}{E_2 E_3}\, \delta^4(p_1 - p_2 - p_3)\, d^3\vec{p}_2\, d^3\vec{p}_3$$

- In general: $\mathcal{M}$ is a function $\vec{p}_2$ and $\vec{p}_3$ and cannot be taken out of the integral!

b.) Scattering:

Suppose particles 1 and 2 collide, producing particles 3, 4, ..., n:

$$1 + 2 \to 3 + 4 + \cdots + n$$

Cross-section:

$$d\sigma = |\mathcal{M}|^2 \frac{\hbar S}{4\cdot\left((p_1\cdot p_2)^2 - (m_1 m_2 c^2)^2\right)^{1/2}}\left[\left(\frac{c\, d^3\vec{p}_3}{(2\pi)^3 2E_3}\right)\left(\frac{c\, d^3\vec{p}_4}{(2\pi)^3 2E_4}\right)\cdots\right.$$

$$\left.\left(\frac{c\, d^3\vec{p}_n}{(2\pi)^3\, 2E_n}\right)\right] \times (2\pi)^4\, \delta^4(p_1 + p_2 - p_3 - p_4 - \cdots - p_n)$$

Note:
- $p_i = (E_i/c, \vec{p}_i)$
- S: statistical factor: $1/j!$ for each group of identical particles

5. Lagrange formalism

In classical mechanics, L is derived $L = T - U$, but in quantum field theory $\mathcal{L}$ is usually taken as axiomatic.

Lagrangian density

$\mathcal{L}$ is a function of fields ϕ_i and their derivatives:

$$\partial_\mu \phi_i \equiv \frac{\partial \phi_i}{\partial x^\mu}$$

function of position and time

Euler-Lagrange equation:

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_i} \qquad i = 1, 2, 3, \ldots}$$

I am now going to present the Lagrangian density $\mathcal{L}$ for three fundamental free field cases:

- Klein-Gordon equation: spin 0
- Dirac equation: spin $\frac{1}{2}$
- Proca equation: spin 1

• Note :

Knowing the Lagrangian allows me to evaluate through :

- canonical quantisation or
- Path integral formalism

a set of Feynman rules for a given to calculate $\mathcal{M}$ and therefore the dynamical part of a cross-section and decay rate.

a.) Klein - Gordon Lagrangian : spin 0

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^2$$ 'free field'

scaler fields

Let's find the Klein - Gordon equation :

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -\left(\frac{mc}{\hbar}\right)^2 \phi$$

Therefore:

$$\boxed{\partial_\mu \partial^\mu \phi + \left(\frac{mc}{\hbar}\right)^2 \phi = 0}$$

Klein - Gordon equation

b.) Dirac equation: spin 1/2

$$\boxed{\mathcal{L} = i(\hbar c)\,\bar{\Psi}\,\gamma^\mu \partial_\mu \Psi - (mc^2)\,\bar{\Psi}\Psi}$$

'spinor field'

4x4 γ matrices

Dirac spinor
4-component element: spin $\frac{1}{2}$ up and spin $\frac{1}{2}$ down

Ψ and adjoint $\bar{\Psi}$ are independent field variables

Dirac equation:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = i\hbar c\,\gamma^\mu \partial_\mu \Psi - mc^2 \Psi$$

$$\boxed{i\gamma^\mu \partial_\mu \Psi - \left(\frac{mc}{\hbar}\right)\Psi = 0}$$ Dirac equation

similarly for adjoint $\bar{\Psi}$.

c.) Proca equation: spin 1

Start now with a vector field, A^μ:

$$\mathcal{L} = -\frac{1}{16\pi}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4\pi}(\partial^\mu A^\nu - \partial^\nu A^\mu) \qquad \frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{1}{4\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu$$

we get:

- Proca-equation: $m \neq 0$, spin 1

$$\boxed{\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0}$$

Short form: $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$

Then:

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu$$

Field equation: $\boxed{\partial_\mu F^{\mu\nu} + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0}$

With $m = 0$ we get: $\partial_\mu F^{\mu\nu} = 0$

6. Examples for phase space integration for various processes:

a.) Decay:

1. Particle of mass m decays into two massless particles: Example $\pi^0 \to \gamma + \gamma$

$1 \to 2 + 3$

matrix: $\mathcal{M}(\vec{p}_2, \vec{p}_3)$

1: at rest

- 4-dim delta-function:

$$\delta^4(p_1 - p_2 - p_3) = \delta\left(mc - \frac{E_2}{c} - \frac{E_3}{c}\right)\delta^3(-\vec{p}_2 - \vec{p}_3)$$

With $m_2 = m_3 = 0$, we have $E_2 = |\vec{p}_2| \cdot c$ and $E_3 = |\vec{p}_3| \cdot c$

$m_1 = m$

Thus:

$$\Gamma = \frac{S}{\hbar m} \cdot \left(\frac{1}{4\pi}\right)^2 \frac{1}{2} \int \frac{|\mathcal{M}|^2}{|\vec{p}_2| \cdot |\vec{p}_3|} \times$$

$$\delta\left(mc - \frac{E_2}{c} - \frac{E_3}{c}\right) \cdot \delta^3(-\vec{p}_2 - \vec{p}_3)\, d^3\vec{p}_2\, d^3\vec{p}_3$$

Do $\vec{p}_3$ integral:

$$\Gamma = \frac{S}{2(4\pi)^2 \hbar m} \int \frac{|\mathcal{M}|^2}{|\vec{p}_2|^2}\, \delta(mc - 2|\vec{p}_2|)\, d^3\vec{p}_2$$

- Go to spherical coordinates:

$$d^3\vec{p}_2 = |\vec{p}_2|^2 \, d|\vec{p}_2| \sin\theta \, d\theta \, d\phi$$

With: $\iint \sin\theta \, d\theta \, d\phi = 4\pi$ we get:

$$\Gamma = \frac{S}{8\pi\hbar m} \int_0^\infty |\mathcal{M}|^2 \, \delta(mc - 2|\vec{p}_2|) \, d|\vec{p}_2|$$

depends only on $|\vec{p}_2|$

π^0 is a scalar!

now use δ-function relation:

$$\delta(kx) = \frac{1}{|k|} \cdot \delta(x)$$

With this we get: $\delta(mc - 2|\vec{p}_2|) =$

$$\delta\left(-2\left(|\vec{p}_2| - \frac{mc}{2}\right)\right) = \frac{1}{2}\,\delta\left(|\vec{p}_2| - \frac{mc}{2}\right)$$

Therefore:

$$\boxed{\Gamma = \frac{S}{16\pi\hbar m}|\mathcal{M}|^2}$$

Note: $\mathcal{M}$ is evaluated at $\vec{p}_3 = -\vec{p}_2$ and $|\vec{p}_2| = \frac{mc}{2}$

$S = \frac{1}{2!} = \frac{1}{2}$ (two identical particles)

2. General two-body decay: $1 \rightarrow 2 + 3$

$m_1 \quad m_2 \quad m_3$

(at rest)

Start with previous expression for Γ before performing last integral:

$$\Gamma = \frac{S}{2(4\pi)^2 \hbar m} \int_0^\infty \frac{|\mathcal{M}|^2}{|\vec{p}_2|^2} \, \delta(mc - 2|\vec{p}_2|) \, d^3\vec{p}_2$$

$$|\vec{p}_2| = \frac{E_2}{c} \qquad (m_2 = 0)$$

Now we have:

$m \rightarrow m_1$

$$|\vec{p}_2| \;\begin{cases} \nearrow & E_2 = c\sqrt{m_2^2 c^2 + \underline{\vec{p}_2^{\,2}}} \\ \searrow & E_3 = c\sqrt{m_3^2 c^2 + \underline{\vec{p}_2^{\,2}}} \end{cases}$$

Therefore:

$$\Gamma = \frac{S}{2(4\pi)^2 \hbar m} \int \frac{|\mathcal{M}|^2 \, \delta\left(m_1 c - \sqrt{m_2^2 c^2 + \vec{p}_2^{\,2}} - \sqrt{m_3^2 c^2 + \vec{p}_2^{\,2}}\right)}{\sqrt{m_2^2 c^2 + \vec{p}_2^{\,2}} \cdot \sqrt{m_3^2 c^2 + \vec{p}_2^{\,2}}} \, d^3\vec{p}_2$$

$|\mathcal{M}|^2$ is only a function of $|\vec{p}_2|$!

Introduce a new variable: $\boxed{\rho = |\vec{p}_2|}$

With : $\delta(m_1 c - E/c) = c\,\delta(E - m_1 c^2)$

$$\boxed{\Gamma = \frac{S\,|\mathcal{M}|^2 p_0}{8\pi\hbar m_1^2 c}}$$

here: $p_0 = p(E = m_1 c^2)$

With $p = |\vec{p}|$ (magnitude of either outgoing particle):

$$\boxed{\Gamma = \frac{S\,|\vec{p}|}{8\pi\hbar m_1^2 c}\,|\mathcal{M}|^2}$$

Note:

This is rather simple! (Two-body decay)
With 3 or more particles in the final state, the functional form of $\mathcal{M}$ has to be known to get the final result!

4.) Two- body scattering: CM frame

$(E_1/c;\ \vec{p}_1)$ $(E_2/c;\ \vec{p}_2)$

$1 + 2 \rightarrow 3 + 4$

CM frame: $\vec{p}_2 = -\vec{p}_1$; $p_1 \cdot p_2 = \frac{E_1 E_2}{c^2} + \vec{p}_1^{\,2}$

Start with:

$$d\sigma = |\mathcal{M}|^2 \frac{\hbar^2 S}{4\left((p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2\right)^{1/2}} \left[\left(\frac{c\, d^3\vec{p}_3}{(2\pi)^3 2E_3}\right) \cdot \left(\frac{c\, d^3\vec{p}_4}{(2\pi)^3 2E_4}\right)\right]$$

$$\times (2\pi)^4 \cdot \delta^4(p_1 + p_2 - p_3 - p_4)$$

- use:

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = (E_1 + E_2)\,|\vec{p}_1|/c \rightarrow HW$$

$$d\sigma = |\mathcal{M}|^2 \frac{\hbar^2 S \cdot (2\pi)^4}{4(E_1 + E_2)\,|\vec{p}_1|/c} \frac{c^2}{(2\pi)^6} \frac{1}{4 E_3 \cdot E_4} d^3\vec{p}_3\ d^3\vec{p}_4$$

$$d\sigma = |\mathcal{M}|^2 \frac{\hbar^2 S\ c^3}{(E_1 + E_2) \cdot |\vec{p}_1|} \cdot \frac{(2\pi)^4}{\underbrace{4 \cdot (2\pi)^6 \cdot 4}_{8^2 \pi^2}} \cdot \frac{1}{E_3 \cdot E_4} d^3\vec{p}_3\ d^3\vec{p}_4$$

then:

$$d\sigma = |\mathcal{M}|^2 \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S\ |\mathcal{M}|^2 c}{(E_1 + E_2) \cdot |\vec{p}_1|} \cdot \frac{d^3\vec{p}_3\ d^3\vec{p}_4}{E_3\ E_4} \delta^4(p_1 + p_2 - p_3 - p_4)$$

re-write now the δ-function:

$$\delta^4(p_1+p_2-p_3-p_4) = \delta\left(\frac{E_1+E_2-E_3-E_4}{c}\right)\delta^3(-\vec{p}_3-\vec{p}_4)$$

use: $E_3 = c\sqrt{m_3^2c^2+\vec{p}_3^{\,2}}$ $\quad E_4 = c\sqrt{m_4^2c^2+\vec{p}_4^{\,2}}$

carry out $\vec{p}_4$ integral: $\vec{p}_4 = -\vec{p}_3$

$$d\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|M|^2 c}{(E_1+E_2)|\vec{p}_1|}\cdot\frac{\delta\left((E_1+E_2)/c-\sqrt{m_3^2c^2+\vec{p}_3^{\,2}}-\sqrt{m_4^2c^2+\vec{p}_3^{\,2}}\right)d^3\vec{p}_3}{\underbrace{\sqrt{m_3^2c^2+\vec{p}_3^{\,2}}}_{E_3/c}\cdot\underbrace{\sqrt{m_4^2c^2+\vec{p}_3^{\,2}}}_{E_4/c}}$$

use sperical coordinates:

$$d^3\vec{p}_3 = \rho^2\,d\rho\,d\Omega \qquad \rho = |\vec{p}_3|$$

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{\hbar}{8\pi}\right)^2\frac{Sc}{(E_1+E_2)\cdot|\vec{p}_1|}\int_0^\infty |M|^2$$

$$\cdot\frac{\delta\left((E_1+E_2)/c-\sqrt{m_3^2c^2+\rho^2}-\sqrt{m_4^2c^2+\rho^2}\right)}{\sqrt{m_3^2\rho^2+\rho^2}\,\sqrt{m_4^2c^2+\rho^2}}\cdot\rho^2\,d\rho$$

M depends on $|\vec{p}_3|$ and θ: $\vec{p}_1\cdot\vec{p}_3 = |\vec{p}_1|\cdot|\vec{p}_3|\cdot\cos\theta$

The integral over ρ is the same as before for the general two-body decay:

Result:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S\,|\mathcal{M}|^2}{(E_1+E_2)^2} \cdot \frac{|\vec{p}_f|}{|\vec{p}_i|}}$$

magnitude of either out-going momenta

magnitude of either in-coming momenta