Chapter 2: Deriving AdS/CFT

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Lecture 16

Important equations for this lecture from the previous ones:

1. The spacetime metric from N D3-branes in IIB SUGRA , equation (13) and (14) in lecture 15:

$$ds^2 = f(r)\Big(-dt^2 + d\vec{x}^2\Big) + h(r)\Big(dr^2 + r^2 d\Omega_e^2\Big) \quad ; \tag{1}$$

$$f(r) = \frac{1}{h(r)} = H^{-1/2}(r) \quad , \quad H(r) = 1 + \frac{R^4}{r}^4 \quad , \quad R^4 = N\frac{4}{\pi^2}G_N T_3 = N4\pi g_s \alpha' \tag{2}$$

2. The relation between the gravitational constant G_N and string theory's g_s and α', equation (15) in lecture 12:

$$G_N = 8\pi^6 g_s^2 \alpha'^2 \tag{3}$$

2.2: D-BRANES AS SPACETIME GEOMETRY (cont.)

From the spacetime metric given in equation (1) and (2), the physical interpretation of R can be seen:

1. For $r \to \infty$, $f(r) = h(r) = 1$, as the spacetime geometry is asymptotically flat.
2. For $r \gg R$, then one arrives at the long-range Coulomb potential $\sim \frac{1}{r^4}$ in $D = 10$ due to a 3D object:

$$f(r) = 1 + \mathcal{O}\Big(\frac{R^4}{r^4}\Big) \quad , \quad h(r) = 1 + \mathcal{O}\Big(\frac{R^4}{r^4}\Big) \tag{4}$$

3. For $r \sim R$, the deformation of spacetime metric from D3-branes become significant, with the curvature $\sim R^{-2}$. In order for $\alpha' R^{-2} \ll 1$ (so that SUGRA is valid), one need $g_s N \gg 1$ and $g_s \ll 1$.
4. For $r \to 0$ as one approaches the D3-branes, then $H(r) \approx \frac{R^4}{r^4}$:

$$ds^2 = \frac{r^2}{R^2}\Big(-dt^2 + d\vec{x}^2\Big) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 \tag{5}$$

The spacetime is now factorized into $AdS_5 \times S^5$, with the S^5 has a constant radius R. Another interesting feature of this metric is that $r = 0$ is now sits at an infinite proper distance away, as the branes seems to be essentially disappeared (no source) and there're only the deformed geometry and F_5 flux in spacetime.

Now, we has 2 descriptions of N D-branes:

1. Description A: D-branes in flat spacetime where open strings can end.

2. Description B: Deformed spacetime metric given in equation (4) with F_5 fluxes on S^5 where only closed strings can propagate.

These 2 descriptions are expected to be equivalent. In priciple, both of them can be extended to be valid for all α' and g_s. This is a surprising statement, but no much can be done about it, since both sides are complicated and not very well known. In 1997, J. Maldacena considered a special limit of this equivalent, the low energy limit (fixed the energy scale E and take $\alpha' \to 0$, or fixed α' and take $E \to 0$), and it is known nowadays as the AdS/CFT correspondence:

1. Description A: Open strings give $\mathcal{N} = 4$ SYM theory with the gauge group $U(N)$ and the Yang-Mills coupling $g_{YM}^2 = 4\pi g_s$, closed strings give graviton and other massless fields, and note that the coupling between massless open and closed strings:

$$G_N \sim g_s^2 \alpha'^4 \tag{6}$$

As $E \to 0$, the $\mathcal{N} = 4$ SYM decouples from gravitons and other closed string modes. Effectively, the theory is that of $\mathcal{N} = 4$ SYM and free gravitons.

2. Description B: From the spacetime metric of N D3-branes, one should be careful with which time to use and define the energy. The energy of D3-branes in description A is defined with t given in equation (1), which is the time at $r = \infty$. At a general value of r, the local proper time $d\tau = H^{-1/4}(r)dt$ so then the local energy $E_\tau = H^{-1/4}E$. For $r \gg R$, $H(r) \approx 1$ and $E^2\alpha' \to 0$, hence all massive string modes decouple. For $r \ll R$, $H(r) \approx \frac{R^4}{r^4}$, and the low energy limit $E^2\alpha' \to 0$ means:

$$E_\tau^2 \frac{r^2}{R^2}\alpha' \to 0 \quad \Rightarrow \quad E_\tau^2 \frac{r^2}{\sqrt{4\pi g_s N}} \to 0 \tag{7}$$

This means, for any E_τ, the low energy limit means $r \to 0$. Which means, for sufficiently small r (close to the D3-branes), any massive stringy modes are allowed. The $r \to 0$ region has $AdS_5 \times S^5$ geometry with full stringy description, so the low energy limit is that of the free gravitons at $r = \infty$ and full string theory (with D-branes, which translational dynamics is actually playing an important role) in $AdS_5 \times S_5$ − these 2 sectors decouple.

Equating description A and B at low energy, one has $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$ (characterized by g_{YM}^2 and N) is equivalent to the full IIB superstring theory in $AdS_5 \times S^5$ (characterized by g_s and $\frac{R^2}{\alpha'}$) with D-branes. With the help from equation (2), one gets the relations:

$$g_{YM}^2 = 4\pi g_s \quad , \quad g_{YM}^2 N = \frac{R^4}{\alpha'^2} \quad , \quad \frac{G_N}{R^8} = \frac{\pi^4}{2N^2} \tag{8}$$

2.3: AdS/CFT DUALITY

2.3.1: AdS SPACETIME

From equation (5), the AdS spacetime metric:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 \tag{9}$$

If $\vec{x}$ is d-dimensional then this metric described AdS_{d+1} spacetime. R is the AdS curvature radius, and r runs from 0 to the boundary ∞. From the general relativity Einstein's field equation point of view, AdS is a spacetime of constant curvature with negative cosmological constant:

$$\mathcal{R}_{MN} - \frac{1}{2}g_{MN}(\mathcal{R} - 2\Lambda) = 0 \quad ; \quad \Lambda < 0 \tag{10}$$

The solution of the given tensor equation:

$$\mathcal{R} = \frac{2(d+1)}{d-1}\Lambda \quad , \quad \Lambda = -\frac{1}{2}d(d-1)\frac{1}{R^2} \rightarrow \mathcal{R} = -d(d+1)R^2 \quad , \quad \mathcal{R}_{MNPQ} = -R^2(g_{MP}g_{NQ} - g_{MQ}g_{NP}) \tag{11}$$

Another convenient choice for coordinates in AdS space is $z = \frac{R^2}{r^2}$, runs from the boundary 0 to ∞:

$$ds^2 = \frac{R^2}{z^2}\Big(- dt^2 + d\vec{x}^2 + dz^2\Big) \tag{12}$$

It should be noted that equation (9) and (12) only cover 1 part of the full AdS spacetime, called the Poincare patch. Indeed, to cover the whole AdS spacetime one needs an infinite number of copies of the Poincare patch. The global AdS_{d+1} spacetime can be described as a hyperboloid in a flat Lorentz spacetime of signature $(2, d)$:

$$X_{-1}^2 + X_0^2 - \vec{X}^2 = R^2 \quad , \quad ds^2 = -dX_{-1}^2 - dX_0^2 + d\vec{X}^2 \tag{13}$$

Let's look more closely to the geometrical structure of AdS space:

1. The Poincare coordinates:

$$r = X_{-1} + X_d \quad , \quad x^\mu = \frac{R}{r}X^\mu \tag{14}$$

Therefore, the coordinates described by equation (9) and (12) only corresponds to the $r > 0$ branch.

2. The global coordinates:

$$X_0 = R\sqrt{1+r^2}\cos\tau \quad , \quad X_{-1} = R\sqrt{1+r^2}\sin\tau \quad , \quad X_0^2 + X_{-1}^2 = R^2(1+r^2) \quad , \quad \vec{X}^2 = R^2r^2 \tag{15}$$

Let τ runs from $-\infty$ to $+\infty$, then:

$$ds^2 = R^2\Bigg(- (1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1}^2\Bigg) \tag{16}$$

For $r = \tan\rho$ with $\rho \in \Big[0, \frac{\pi}{2}\Big]$:

$$ds^2 = \frac{R^2}{\cos^2\rho}\Big(- d\tau^2 + d\rho^2 + \sin^2\rho d\Omega_{d-1}^2\Big) \tag{17}$$

This choice of coordinates has the AdS center at $\rho = 0$ and the AdS boundary at $\rho = \frac{\pi}{2}$, and the geometry of the boundary is $S^{d-1} \times \mathbb{R}$.

The spacetime interval in the boundary can be calculated with:

$$ds_{boundary}^2 \sim -d\tau^2 + d\Omega_{d-1}^2 \tag{18}$$

It takes a light ray $\tau = \frac{\pi}{2}$ to reach the boundary, but a massive particle can never reach the boundary since at some point it will turned back by gravitational pull. The AdS spacetime is like a confining box of size $\sim R$.

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