8.821/8.871 Holographic duality

MIT OpenCourseWare Lecture Notes

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Lecture 7

In fact, any orientable two dimensional surface is classified topologically by an integer h, called the genus. The genus is equal to the number of "holes" that the surface has (Fig. 1).

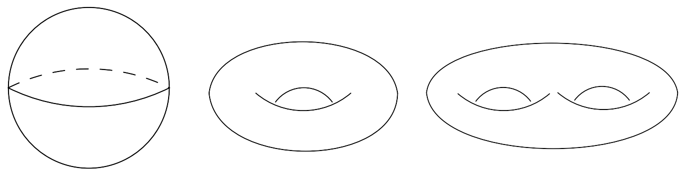

Figure 1: sphere (genus-0), torus (genus-1) and double torus(genus-2).

An topological invariant of the manifold is the Euler character:

$$\chi = 2 - 2h$$

Here we make some claims:

1. For any non-planar diagram, there exists an integer h, such that the diagram can be straightened out (*i.e.* non-crossing) on a genus-h surface, but not on a surface with a smaller genus.
2. For any non-planar diagram, the power of N that comes from contracting propagators is given by the number of faces on such a genus-h surface, *i.e.* the number of disconnected regions separated by the diagram.

Both claims are self-evident after a bit practices.

In general, a vacuum diagram has the following dependence on g^2 and N:

$$A \sim (g^2)^E (g^2)^{-V} N^F$$

where E is the number of propagators, V is the number of vertices, F is the number of faces. This does not give a sensible $N \to \infty$ limit or $1/N$ expansion, since there is no upper limit on F. However, 't Hooft suggests that we can take the limit $N \to \infty$ and $g^2 \to 0$ but keep $\lambda = g^2 N$ fixed. Then

$$A \sim (g^2 N)^{E-V} N^{F+V-E} = \lambda^{L-1} N^{\chi} = \lambda^{L-1} N^{2-2h}$$

where L is the number of loops. The relation $\chi = F + V - E$ is guaranteed by the following theorem.

Theorem: Given a surface composed of polygons with F faces, E edges and V vertices, the Euler character satisfy

$$\chi = F + V - E = 2 - 2h$$

Since each Feynman diagram can be considered as a partition of the surface separating it into polygons, then the above theorem also works for our counting in N.

Thus in this limit, to the leading order in N is the planar diagrams

$$N^2(c_0 + c_1\lambda + c_2\lambda^2 + \cdots) = N^2 f_0(\lambda)$$

Because $\log Z$ evaluates the sum of all vacuum diagrams, we can conclude, including higher order $1/N^2$ corrections:

$$\log Z = \sum_{h=0}^{\infty} f_h(\lambda) = N^2 f_0(\lambda) + f_1(\lambda) + \frac{1}{N^2} f_2(\lambda) + \cdots$$

The first term comes from the planar diagrams, second term from the genus-1 diagrams, etc.

There is a heuristic way to understand $\log Z = O(N^2) + \cdots$. Since $Z = \int D\Phi e^{iS[\Phi]}$ and we can rewrite the Lagrangian as

$$\mathcal{L} = \frac{N}{\lambda}\operatorname{Tr}\left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{4}\Phi^4\right]$$

The trace also gives a factor of N, thus $\mathcal{L} \sim O(N^2)$, we have $\log Z \sim O(N^2)$.

Clearly our discussion only depends on the matrix nature of the fields. So for any Lagrangian of matrix valued fields of the form

$$\mathcal{L} = \frac{N}{\lambda}\operatorname{Tr}(\cdots)$$

we would have

$$\log Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda)$$

To summarize, in the 't Hooft limit, $1/N$ expansion is the same as topological expansion in terms of topology of Feynman diagrams.

General observables

Now we have introduced two theories:

$$(a) \qquad \mathcal{L} = -\frac{1}{g^2}\operatorname{Tr}\left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{4}\Phi^4\right]$$
$$(b) \qquad \mathcal{L} = \frac{1}{g_{YM}^2}\left[-\frac{1}{4}\operatorname{Tr}F_{\mu\nu}F^{\mu\nu} - i\overline{\Psi}(\not{D} - m)\Psi\right]$$

(a) is invariant under the global $U(N)$ transformation: $\Phi \to U\Phi U^\dagger$ with U constant $U(N)$ matrix, *i.e.* the theory has a global $U(N)$ symmetry. (b) is invariant under local $U(N)$ transformation:
$A_\mu \to U(x)A_\mu U^\dagger(x) - i\partial_\mu U(x)U^\dagger(x)$ with $U(x)$ any $U(N)$ matrix, the theory has a $U(N)$ gauge symmetry.

On the other hand, consider allowed operators in the two theories. In (a), operators like ${\Phi^a}_b$ are allowed, although it is not invariant under global $U(N)$ symmetry. But in (b), allowed operators must be gauge invariant, so ${\Phi^a}_b$ is not allowed. So if we consider gauge theories: $\mathcal{L} = \mathcal{L}(A_\mu, \Phi, \cdots)$, the allowed operators will be

$$\text{Single-trace operators} : \operatorname{Tr}(F_{\mu\nu}F^{\mu\nu}), \operatorname{Tr}(\Phi^n), \cdots$$
$$\text{Multiple-trace operators} : \operatorname{Tr}(F_{\mu\nu}F^{\mu\nu})\operatorname{Tr}(\Phi^2), \operatorname{Tr}(\Phi^2)\operatorname{Tr}(\Phi^n)\operatorname{Tr}(\Phi^n), \cdots$$

We denote single-trace operators as $\mathcal{O}_k$, $k = 1, \cdots$ represents different operators. Then multiple-trace ones will be like $\mathcal{O}_m\mathcal{O}_n(x)$, $\mathcal{O}_{m_1}\mathcal{O}_{m_2}\mathcal{O}_{m_3}(x), \cdots$

So general observables will be correlation functions of gauge invariant operators, here we focus on local operators:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c \tag{1}$$

Note that it is enough to focus on single-trace operators since multiple-trace ones are products of them. Since we are working in the t' hooft limit, we want to know how correlation (Eq. 1) scales in the large N limit. There is a trick, consider

$$Z[J_1, \cdots, J_n] = \int DA_\mu D\Phi \cdots \exp(iS_{eff}) = \int DA_\mu D\Phi \cdots \exp\left[iS_0 + iN\sum_j \int J_i(x)\mathcal{O}_i(x)\right]$$

Then the correlation (Eq. 1) can be expressed as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c = \frac{\delta^n \log Z}{\delta J_1(x_1)\cdots\delta J_n(x_n)}|_{J_1=\cdots=J_n=0}\frac{1}{(iN)^n} \tag{2}$$

With $\mathcal{O}_i$ single-trace operators, S_{eff} has the form $N\,\mathrm{Tr}\,(\cdots)$. So we have

$$\log Z\left[J_1,\cdots,J_n\right]=\sum_{h=0}^{\infty}N^{2-2h}f_h(\lambda,\cdots)$$

Applying Eq. (2),

$$\langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle_c\sim N^{2-n}\left[1+O(\frac{1}{N^2})+\cdots\right]$$

e.g.

$$\begin{aligned}\langle\mathbb{1}\rangle &\sim O(N^2)+O(N^0)+\cdots\\ \langle\mathcal{O}\rangle &\sim O(N)+O(N^{-1})+\cdots\\ \langle\mathcal{O}_1\mathcal{O}_2\rangle_c &\sim O(N^0)+O(N^{-2})+\cdots\\ \langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_c &\sim O(N^{-1})+O(N^{-3})+\cdots\end{aligned}$$

All leading order contributions come from planar diagrams.

Physical implications:

1. In the large N limit, $\mathcal{O}(x)|0\rangle$ can be interpreted as creating a single-particle state ("glue ball"). Similarly $:\mathcal{O}_1\cdots\mathcal{O}_n(x):|0\rangle$ represents n-particle state.
   - since $\langle\mathcal{O}_i\mathcal{O}_j\rangle\sim O(N^0)$, we can diagonalize them such that $\langle\mathcal{O}_i\mathcal{O}_j\rangle\propto\delta^i{}_j$.
   - $\langle\mathcal{O}_i(x)\mathcal{O}_j^2(y)\rangle\sim O(N^{-1})\to 0$ as $N\to\infty$, *i.e.* there is no mixing between single-trace and multiple-trace operators in the large N limit.
   - $\langle\mathcal{O}_1\mathcal{O}_2(x)\mathcal{O}_1\mathcal{O}_2(y)\rangle=\langle\mathcal{O}_1(x)\mathcal{O}_1(y)\rangle\langle\mathcal{O}_2(x)\mathcal{O}_2(y)\rangle+\langle\mathcal{O}_1\mathcal{O}_2(x)\mathcal{O}_1\mathcal{O}_2(y)\rangle_c$, the first term is the multiple of independent propagators of $\mathcal{O}_1$ and $\mathcal{O}_2$ states, the second term scales like $O(N^{-2})$.

   Note that it is not necessary there exists a stable on-shell particle associated with $\mathcal{O}_i(x)|0\rangle$.

2. The fluctuations of "glue balls" are suppressed:
   suppose $\langle\mathcal{O}\rangle\neq 0\sim O(N)$, the variance of $\langle\mathcal{O}\rangle$ is $\langle\mathcal{O}^2\rangle-\langle\mathcal{O}\rangle^2=\langle\mathcal{O}^2\rangle_c\sim O(1)$, *i.e.* $\frac{\sqrt{\langle\mathcal{O}^2\rangle_c}}{\langle\mathcal{O}\rangle}\sim N^{-1}\to 0$. Also $\langle\mathcal{O}_1\mathcal{O}_2\rangle=\langle O_1\rangle\langle O_2\rangle+\langle\mathcal{O}_1\mathcal{O}_2\rangle_c$, the first term scales as $O(N^2)$ while the second term scales as $O(1)$. Thus in the large N limit, it is more like a classical theory.

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Fall 2014