8.821 String Theory
Fall 2008

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Department of Physics
String Theory (8.821) – Prof. J. McGreevy – Fall 2008

Problem Set 4
Fields in AdS

1. **One more symmetry.**

   Define $x^A \equiv (z, x^\mu)$, so that the Poincaré patch metric takes the form

   $$ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2} = \frac{dx^A dx_A}{z^2}.$$

   Show that this metric is preserved by the following *inversion* symmetry:

   $$x^A \to \frac{x^A}{x^B x_B}$$

   i.e. $z \to \frac{z}{z^2+\vec{x}^2}$, $x^\mu \to \frac{x^\mu}{z^2+\vec{x}^2}$. This is a conformal transformation which is not continuously connected to the identity.

   [Bonus question: what does this do to the region of the global AdS space covered by the Poincaré patch coordinates?]

2. **Schrödinger description of AdS instabilities.**[1]

   a) Consider the scalar wave equation in poincaré AdS, in momentum space:

   $$0 = (\Box - m^2)\phi,$$

   with $\phi(x,z) = e^{ik\cdot x}\phi(z)$ By redefining the independent variable, rewrite the scalar wave equation as a Schödinger equation:

   $$(-\partial_z^2 + V(z))\Psi(z) = \omega^2 \Psi(z)$$

   with ω^2 playing the role of energy. (Hint: Rewrite $\Psi(z) = A(z)\phi(z)$, and choose the function A to set to zero the term in the wave equation multiplying $\Psi'(z)$.)

[1] I learned about this trick from Sean Hartnoll. If you get stuck or want to see a recent application of this technique, see *e.g.* appendix A of 0810.1563.

b) A property of a field configuration which generally determines whether it is allowed to participate in the physics (*i.e.* whether it is fixed by boundary conditions, or whether it can happen on its own) is whether it is *normalizable.* In Euclidean space, this generally just means that the euclidean action evaluated on the configuration is finite, since then it contributes to the path integral with a finite Boltzmann factor $e^{-S[\phi]}$. In Minkowski signature, a field configuration should be considered normalizible if it has finite energy; for a scalar field, this energy is

$$\mathcal{E}[\phi] \equiv \int_\Sigma T_{AB} \xi^A n^B$$

where the integral is over some fixed-time slice Σ, n is a unit normal vector to Σ, ξ is a time-like killing vector, and T_{AB} is the stress tensor for the bulk field, $T_{AB} \equiv \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{AB}}$.

Given the redefinition you found in part a), what is the relationship between normalizable wave functions in the usual QM sense (*i.e.* $||\Psi||^2 < \infty$) and normalizable solutions of the AdS wave equation (*i.e.* those with finite energy)?

c) Set $\vec{k} = 0$. Show that when m^2 passes through the value $-|m^2_{BF}|$ from above, this Schrödinger equation develops a single normalizable negative-energy state.[2] (Recall that the BF bound is $m^2 \geq m^2_{BF} = -\frac{D^2}{4L^2}$.) This corresponds by the map above to a normalizable mode with *imaginary* frequency – *i.e.* a linear instability.

3. **Dimensions of vector operators.**

   By studying the boundary behavior of solutions to the bulk Proca equations, derived from the action

$$S_{\text{bulk}}[A] = -\kappa \int_{AdS} d^{p+2}x \sqrt{g} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A^\mu A_\mu \right)$$

   compute the scaling dimension Δ of the current coupled to a bulk vector field A_μ of arbitary mass:

$$S_{\text{bdy}} = \int_{\partial AdS} d^{p+1}x \sqrt{\gamma} A_\mu J^\mu.$$

   For purposes of this problem, define Δ to be the power of z of the *subleading* power-law solution of the bulk equation near the boundary at $z = 0$.

[2] If you want to learn more about potentials of this form, see H. Hammer and B. Swingle, Annals Phys. 321 (2006) 306-317 [arXiv:quant-ph/0503074v1].

4. **Saturating the unitarity bound.**[3]

For this problem we will work in Euclidean AdS. Feel free to redo the problem for the Lorenzian case (using the notion of normalizability described in problem two).

a) Consider the usual bulk action for a (free) scalar field in AdS_{D+1}:

$$S_{\text{usual}}[\phi] = -\frac{1}{2}\int d^D x dz \sqrt{g}\left((\partial\phi)^2 + m^2\phi^2\right)$$

Show that for $m^2L^2 > -\frac{D^2}{4} + 1$,

$$S_{\text{usual}}[z^{\Delta_-}] = \infty.$$

That is, the integral over the radial direction in this action diverges near $z = 0$ when evaluated on a solution of the form $\phi \sim z^{\Delta_-}$, where Δ_- is the *smaller* root of $\Delta(\Delta - D) = L^2m^2$; such a solution is non-normalizable. On the other hand, show that z^{Δ_+} gives a finite value, and hence a solution with these asymptotics is normalizable. With this action, show that the smallest conformal dimension $\Delta = \Delta_+$ of a scalar operator in the dual theory is $\Delta \geq \frac{D}{2}$.

b) Show that the following modified action

$$S_{KW}[\phi] \equiv -\frac{1}{2}\int d^{D+1}x\sqrt{g}\left(\phi(-\Box + m^2)\phi\right) = S_{\text{usual}}[\phi] - \int_{\partial AdS}\sqrt{\gamma}\phi\partial_n\phi.$$

produces the same bulk equations, but has the following property: in the window of mass values (we could call this the KW window)

$$-\frac{D^2}{4} < L^2m^2 < -\frac{D^2}{4} + 1,$$

both roots $\Delta_\pm$ give normalizable solutions with respect to the action S_{KW} (*i.e.* $S_{KW}[z^{\Delta_\pm}] < \infty$).

What is the lowest conformal dimension you can obtain now? (Hint: you can now consider Δ_+ to be the dimension of the source, and Δ_- to be the dimension of the operator.)

[3]This result is from Klebanov-Witten, hep-th/9905104.