8.821 String Theory
Fall 2008

8.821 F2008 Lecture 13: Masses of fields and dimensions of operators

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In today's lecture we will talk about:

1. AdS wave equation near the boundary.
2. Masses and operator dimensions: $\Delta(\Delta - D) = m^2 L^2$.

Erratum: The massive geodesic equation $\ddot{x} + \Gamma \dot{x}\dot{x} = 0$ assumes that the dot differentiates with respect to proper time.

Recap: Consider a scalar in AdS_{p+2} (where $p+1$ is the number of spacetime dimensions that the field theory lives in). Let the metric be:

$$ds^2 = L^2 \frac{dz^2 + dx^\mu dx_\mu}{z^2}, \tag{1}$$

then the action takes the form:

$$S[\phi] = -\frac{\kappa}{2} \int d^{p+1}x \sqrt{g} \left((\partial\phi)^2 + m^2\phi^2 + b\phi^3 + ... \right), \tag{2}$$

where $(\partial\phi)^2 \equiv g^{AB}\partial_A\phi\partial_B\phi$ and $x^A = (z, x^\mu)$. Our goal is to evaluate:

$$\ln\langle \exp^{-\int d^D x \, \phi_0 \, O} \rangle_{CFT} = \text{extremum}_{[\phi \, | \, \phi \to \phi_0 \ at \ z=\epsilon]} S[\phi], \tag{3}$$

where $S[\phi] \equiv S[\phi^*(\phi_0)] \equiv W[\phi_0]$, i.e. by using the solution to the equation of motion subject to boundary conditions. Now Taylor expand:

$$W[\phi_0] = W[0] + \int d^D x \ \phi_0(x) G_1(x) + \frac{1}{2} \int\int d^D x_1 d^D x_2 \, \phi_0(x_1)\phi_0(x_2) G_2(x_1, x_2) + ... \tag{4}$$

where

$$G_1(x) = \langle O(x) \rangle = \frac{\delta W}{\delta\phi_0(x)}|_{\phi_0=0}, \tag{5}$$

$$G_2(x) = \langle O(x_1)O(x_2) \rangle_c = \frac{\delta^2 W}{\delta\phi_0(x_1)\delta\phi_0(x_2)}|_{\phi_0=0}. \tag{6}$$

Now if there is no instability, then ϕ_0 is small and so is ϕ, so you can ignore third order terms in ϕ. From last time:

$$S[\phi] = \frac{\kappa}{2} \int_{AdS_{p+2}} d^{p+2}x \, \sqrt{g} \, [\phi \, (-\nabla^2 + m^2) \, \phi + \mathcal{O}(\phi^3)] - \frac{\kappa}{2} \int_{\partial AdS} d^{p+1}x \, \sqrt{\gamma} \, \phi \, (n.\partial) \, \phi, \tag{7}$$

where the last term is the boundary action, n is a normalized vector perpendicular to the boundary and

$$\nabla^2 = \frac{1}{\sqrt{g}} \partial_A (\sqrt{g} g^{AB} \partial_B). \tag{8}$$

Now if the scalar field satisfies the wave equation:

$$(-\nabla^2 + m^2)\phi^* = 0, \tag{9}$$

$$W[\phi_0] = S_{bdy}[\phi^*[\phi_0]], \tag{10}$$

then we can use translational invariance in $p+1$ dimensions, $x^\mu \to x^\mu + a^\mu$, in order to Fourier decompose the scalar field:

$$\phi(z, x^\mu) = e^{ik.x} f_k(z). \tag{11}$$

Now, substituting (11) into (9) and assuming that the metric only depends on z we get:

$$\begin{aligned} 0 &= (g^{\mu\nu} k_\mu k_\nu - \frac{1}{\sqrt{g}} \partial_z (\sqrt{g} g^{zz} \partial_z) + m^2) f_k(z) && (12) \\ &= \frac{1}{L^2} [z^2 k^2 - z^{D+1} \partial_z (z^{-D+1} \partial_z) + m^2 L^2] f_k, && (13) \end{aligned}$$

where we have used $g^{\mu\nu} = (z/L)^2 \delta^{\mu\nu}$. The solutions of (12) are Bessel functions but we can learn a lot without using their full form. For example, look at the solutions near the boundary (i.e. $z \to 0$). In this limit we have power law solutions, which are spoiled by the $z^2 k^2$ term. Try using $f_k = z^\Delta$ in (12):

$$\begin{aligned} 0 &= k^2 z^{2+\Delta} - z^{D+1} \partial_z (\Delta z^{-D+\Delta}) + m^2 L^2 z^\Delta && (14) \\ &= (k^2 z^2 - \Delta(\Delta - D) + m^2 L^2) z^\Delta, && (15) \end{aligned}$$

and for $z \to 0$ we get:

$$\Delta(\Delta - D) = m^2 L^2 \tag{16}$$

The two roots for (16) are

$$\Delta_\pm = \frac{D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 + m^2 L^2}. \tag{17}$$

Comments

- The solution proportional to z^{Δ_-} is bigger near $z \to 0$.
- $\Delta_+ > 0 \ \forall \ m$, therefore z^{Δ_+} decays near the boundary.
- $\Delta_+ + \Delta_- = D$.

Next, we want to improve the boundary conditions that allow solutions, so take:

$$\phi(x,z)|_{z=\epsilon} = \phi_0(x,\epsilon) = \epsilon^{\Delta_-}\phi_0^{Ren}(x), \tag{18}$$

where ϕ_0^{Ren} is the renormalized field. Now with this boundary condition, $\phi(z,x)$ is finite when $\epsilon \to 0$, since ϕ_0^{Ren} is finite in this limit.

Wavefunction renormalization of O (Heuristic but useful)

Suppose:

$$S_{bdy} \ni \int_{z=\epsilon} d^{p+1}x \sqrt{\gamma_\epsilon}\, \phi_0(x,\epsilon)\, O(x,\epsilon) \tag{19}$$

$$= \int d^D x \left(\frac{L}{\epsilon}\right)^D (\epsilon^{\Delta_-}\phi_0^{Ren}(x))\, O(x,\epsilon), \tag{20}$$

where we have used $\sqrt{\gamma} = (L/\epsilon)^D$. Demanding this to be finite as $\epsilon \to 0$ we get:

$$O(x,\epsilon) \sim \epsilon^{D-\Delta_-} O^{Ren}(x) \tag{21}$$

$$= \epsilon^{\Delta_+} O^{Ren}(x), \tag{22}$$

where in the last line we have used $\Delta_+ + \Delta_- = D$. Therefore, the scaling of O^{Ren} is $\Delta_+ \equiv \Delta$.

Comments

- We will soon see that $\langle O(x) O(0)\rangle \sim \frac{1}{|x|^{2\Delta}}$.
- We had a second order ODE, therefore we need two conditions in order to determine a solution (for each k). So far we have imposed:
  1. For $z \to \epsilon$, $\phi \sim z^{\Delta_-}\phi_0 +$ (terms subleading in z). Now we will also impose
  2. ϕ regular in the interior of AdS (i.e. at $z \to \infty$).

Comments on Δ

1. The ϵ^{Δ_-} factor is independent of k and x, which is a consequence of a local QFT (this fails in exotic examples).

2. Relevantness: Since $m^2 > 0 \implies \Delta \equiv \Delta_+ > D$, so O_Δ is an irrelevant operator. This means that if you perturb the CFT by adding O_Δ to the Lagrangian, then:

$$\Delta S = \int d^D x \, (\text{mass})^{D-\Delta} O_\Delta, \tag{23}$$

where the exponent is negative, so the effects of such an operator go away in the IR. For example, consider a dilaton mode with $l > 0$, its mass is given by (for $D = 4$):

$$m^2 = \frac{(l+4)l}{L^2}. \tag{24}$$

The operator corresponding to this is:

$$\text{tr}(F^2 X^{i_1 \cdots i_l}), \tag{25}$$

with $\Delta = 4 + l > D$, therefore it is an irrelevant operator. Now consider a dilaton mode with $l = 0$: then $m^2 = 0$, therefore, $\Delta = D$ and hence it corresponds to a marginal operator (an example of such operator is the Lagrangian). If $m^2 < 0$, then $\Delta < D$, so it corresponds to a relevant operator, but it is ok if m^2 is not too negative ("Breitenlohner - Freedmasn (BF) - allowed tachyons" with $-|m_{BF}|^2 \equiv -(D/2L)^2 < m^2$).

3. Instability: This occurs when a renormalizable mode grows with time without a source. But in order to have $S[\phi] < \infty$, the solution must fall off at the boundary. This requires a gradient energy that $\sim \frac{1}{L}$. Note:

$$\Delta_\pm = \frac{D}{2} \pm \sqrt{\left(\frac{D}{2}\right)^2 + m^2 L^2}. \tag{26}$$

If:

$$m^2 L^2 < (\frac{D}{2})^2 \equiv -|m_{BF}|^2, \tag{27}$$

then $\Delta_\pm$ is complex, therefore we have $\Delta_- = D/2$, which is larger than the unitary bound. In this case, $\phi \sim z^{\Delta_-}$ decays near the boundary (i.e. in the UV). In order to see the instability that occurs when $m^2 L^2 < (\frac{D}{2})^2$ more explicitly, rewrite (9) as a Schrodinger equation, by writing $\phi(z) = A(z)\psi(z)$, where we choose $A(z)$ in order to remove the first derivative of $\psi(z)$. Then, equation (9) becomes:

$$(-\partial_z^2 + V(z))\psi(z) = E\psi(z), \tag{28}$$

where $E = \omega^2 - k^2$, $V(z) = \sigma/z^2$ and $\sigma = m^2 L^2 - (D^2 - 1)/4$. An instability occurs when $E < 0$, i.e. $\omega^2 < 0$ and hence $\phi \sim e^{i\omega t}\phi(z) = e^{+|\omega| t}\phi(z)$ grows with time. Now the claim is that $V = \sigma/z^2$ has no negative energy states if $\sigma > -1/4$. Note that the notion of normalizability here and before are related (Pset 4):

$$||\psi||^2 = \int dz \, \psi^\dagger \psi < \infty, \tag{29}$$

$$\text{and } S[\phi] = \int dz \, \sqrt{g} \, ((\partial\phi)^2 + m^2) \tag{30}$$

4. The formula we found before (expression (16)) depends on the spin. For a $j-$form in AdS we have:

$$(\Delta + j)(\Delta + j - D) = m^2 L^2. \tag{31}$$

For example, for A_μ massless we have:

$$\Delta(j^\mu) = D - 1 \;\; \rightarrow \;\; \text{conserved}, \tag{32}$$

for $g_{\mu\nu}$ massless we have:

$$\Delta(T^{\mu\nu}) = D \;\; \rightarrow \;\; \text{required from CFT}. \tag{33}$$