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8.821 String Theory
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8.821 F2008 Lecture 14: Wave equation in AdS, Green's function

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Topics for this lecture

- Find $\phi^{[\phi_0]}(x)$ by Green functions in x -space (efficient)
- Compute $\langle \mathcal{O} \mathcal{O} \rangle$, counter terms
- Redo in p -space (general)

References

- Witten, hep-th/9802150
- GKP, hep-th/9802109

Solving Wave Equation I (Witten's method)

Let's study the wave equation in AdS in some detail. This first method uses a trick by Witten which is efficient but slightly obscure.

If we know "bulk-to-boundary" Green's function K regular in the bulk, such that

$$(-\square + m^2)K_p(z, x) = 0 \tag{1}$$

$$K_p(z, x) \rightarrow \epsilon^{\Delta-} \delta_\epsilon^D(x - p), \quad z \rightarrow \epsilon \tag{2}$$

where p is some point on the boundary,
then the field in the bulk

$$\phi^{[\phi_0]}(z, x) = \int d^D x' \phi_0^{\text{Ren}}(x') K_{x'}(z, x) \rightarrow z^{\Delta-} \phi_0^{\text{Ren}}(x)$$

solves (1).

Euclidean AdS

Recall the metric on AdS with curvature scale L in the upper half plane coordinates:

$$ds^2 = L^2 \frac{dz^2 + dx^2}{z^2}$$

Now here comes some fancy tricks, thanks to Ed:

Trick (1): Pick $p = \text{“point at } \infty\text{”}$. This implies that the Green’s function $K_\infty(z, x)$ is x -independent.

The wave equation at $k = 0$:

$$0 = [-z^{D+1} \partial_z z^{-D+1} \partial_t + m^2] K_\infty(z)$$

can easily be solved. The solution is power law (recall that in the general- k wave equation, it was the terms proportional to k^2 that ruined the power-law behavior away from the boundary)

$$K_\infty(z) = c_+ z^{\Delta_+} + c_- z^{\Delta_-}$$

We can eliminate one of the constants: $c_- = 0$, whose justification will come with the result.

Trick (2): Use AdS isometries to map $p = \infty$ to finite x . Let $x^A = (x^\mu, z)$, take $x^A \rightarrow (x')^A = x^A / (x^B x_B)$. The inversion of this mapping is:

$$I : \begin{cases} x^\mu \rightarrow \frac{x^\mu}{z^2 + x^2} \\ z \rightarrow \frac{z}{z^2 + x^2} \end{cases}$$

Claim: I

A) is an isometry of AdS (also Minkowski version, see pset 4)

B) is not connected to $\mathbf{1}$ in $\text{SO}(D, 2)$

C) maps $p = \infty$ to $x = 0$, i.e., $I : K_\infty(z, x) \rightarrow K_\infty(z', x') = K_0(z, x) = c_+ z^{\Delta_+} / (z^2 + x^2)^{\Delta_+}$.

Some notes:

(i) That this solves the wave equation (1) as necessary can be checked directly.

(ii) The Green’s function is

$$K_{x'}(z, x) = c_+ \frac{z^{\Delta_+}}{(z^2 + (x - x')^2)^{\Delta_+}} \equiv K(z, x; x')$$

(iii) The limit of the Green’s function as $z \rightarrow 0$, i.e. the boundary is

$$K(z, x; x') \rightarrow \begin{cases} cz^{\Delta_+} \rightarrow 0, & \text{if } x \neq x' \\ cz^{-\Delta_+} \rightarrow \infty, & \text{if } x = x' \end{cases}$$

(recall that $\Delta_+ > 0$ for any D, m). More specifically, the Green's function approaches a delta function:

$$K(z, x; x') \rightarrow \text{const} \cdot \epsilon^{\Delta_-} \delta^D(x - x').$$

Clearly it has support only near $x = x'$, but to check this claim we need to show that it has finite measure:

$$\begin{aligned} \int d^D x \epsilon^{-\Delta_-} K_0(\epsilon, x) &= \int d^D x \frac{c \epsilon^{\Delta_+ - \Delta_-}}{(\epsilon^2 + x^2)^{\Delta_+}} \\ &= \frac{c \epsilon^D \epsilon^{2\Delta_+ - D}}{\epsilon^{2\Delta_+}} \int d^D \bar{x} \frac{1}{(1 + \bar{x}^2)^{\Delta_+}} \\ &= c \frac{\pi^{\frac{D}{2}} \Gamma(\Delta_+ - \frac{D}{2})}{\Gamma(\Delta_+)}. \end{aligned}$$

We will choose the constant c to set this last expression equal to one. Hence,

$$\begin{aligned} \phi^{[\phi_0]}(z, x) &= \int d^D x' K_{x'}(z, x) \phi_0^{\text{Ren}}(x') \\ &= \int d^D x' c \frac{x^{\Delta_+}}{(z^2 + (x - x')^2)^{\Delta_+}} \phi_0^{\text{Ren}}(x'); \end{aligned}$$

this solves (1) and approaches $\epsilon^{\Delta_-} \phi_0^{\text{Ren}}(x)$ as $z \rightarrow \epsilon$.

The action is related to expectation values of operators on the boundary:

$$\begin{aligned} S[\phi^{[\phi_0]}] &= -\ln \langle e^{-\int \phi_0 \mathcal{O}} \rangle \\ &= -\frac{\eta}{2} \int_{\partial \text{AdS}} \sqrt{\gamma} \phi n \cdot \partial \phi \\ &= -\frac{\eta}{2} \int d^D x \sqrt{g} g^{zz} \phi(z, x) \partial_z \phi(z, x) \Big|_{z=\epsilon} \\ &= -\frac{\eta}{2} \int d^D x_1 d^D x_2 \phi_0^{\text{Ren}}(x_1) \phi_0^{\text{Ren}}(x_2) \mathcal{F}_\epsilon(x_1, x_2) \end{aligned}$$

where the ‘‘flux factor’’ is

$$\mathcal{F}_\epsilon(x_1, x_2) \equiv \int d^D x \frac{K(z, x; x_1) z \partial_z K(z, x; x_2)}{z^D} \Big|_{z=\epsilon}.$$

The boundary behavior of K is:

$$K^{\Delta_+}(z, x; x') \Big|_{z=\epsilon} = \epsilon^{\Delta_-} (\delta_\epsilon^D(x - x') + \mathcal{O}(\epsilon^2)) + \epsilon^{\Delta_+} \left(\frac{c}{(x - x')^{2\Delta_-}} + \mathcal{O}(\epsilon^2) \right)$$

the first terms sets: $c^{-1} = \pi^{\frac{D}{2}} \Gamma(\Delta_+ - \frac{D}{2}) / \Gamma(\Delta_+)$, the second term is subleading in z .

$$z \partial_z K(z, x; x') \Big|_{z=\epsilon} = \Delta_+ \epsilon^{\Delta_-} \delta(x - x') + \Delta_+ c z^{\Delta_+} \frac{1}{(x - x')^{2\Delta_+}} + \dots$$

Ok, now for the 2-point correlation function on the boundary:

$$G_2(x_1, x_2) \equiv \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_c = \frac{\delta}{\delta \phi_0(x_1)} \frac{\delta}{\delta \phi_0(x_2)} \left(-S[\phi^{[\phi_0]}] \right) = \eta \mathcal{F}_\epsilon(x_1, x_2).$$

We must be careful when evaluating the cases $x_1 \neq x_2$ and $x_1 = x_2$, which we do in turn.

Firstly, if $x_1 \neq x_2$:

$$\begin{aligned} G_2(x_1 \neq x_2) &= \frac{\eta}{2} \int_{z=\epsilon} d^D x z^{-D} (z^{\Delta_-} \delta^D(x - x_1) + \mathcal{O}(z^2)) \left((\text{ignore by } x_1 \neq x_2) + \frac{\Delta_+ c z^{\Delta_+}}{(x_1 - x_2)^{2\Delta_+}} + \mathcal{O}(z^2) \right) \\ &= \frac{\eta}{2} c \Delta_+ \epsilon^{-D+\Delta_-+\Delta_+} \frac{1}{(x_1 - x_2)^{2\Delta_+}} + \mathcal{O}(\epsilon^2) \\ &= \frac{\eta c \Delta_+}{2(x_1 - x_2)^{2\Delta_+}}. \end{aligned}$$

Good. This is the correct form for a two point function of a conformal primary of dimension Δ_+ in a CFT; this is a check on the prescription.

Secondly, if $x_1 = x_2$:

$$G_2(x_1, x_2) = \eta \left(\Delta_- \epsilon^{2\Delta_- - D} \delta^D(x_1 - x_2) + \frac{c \Delta_+}{(x_1 - x_2)^{2\Delta_+}} + \Delta_+ c^2 \epsilon^{2\Delta_+ - D} \int d^D x \frac{1}{(x - x_1)^{2\Delta_+} (x - x_2)^{2\Delta_+}} \right)$$

As $\epsilon \rightarrow 0$, the first term is divergent, the second term is finite, and the third term vanishes. The first term is called a ‘‘divergent contact term’’. It is scheme-dependent and useless.

Remedy: Holographic Renormalization. Add to S_{geometry} the contact term

$$\begin{aligned} \Delta S = S_{\text{c.t.}} &= \frac{\eta}{2} \int_{\text{bdy}} d^D x \left(-\Delta_- \epsilon^{2\Delta_- - D} (\phi_0^{\text{Ren}}(x))^2 \right) \\ &= -\Delta_- \frac{\eta}{2} \int_{\partial \text{AdS}, z=\epsilon} \sqrt{\gamma} \phi^2(z, x). \end{aligned}$$

Note that this doesn’t affect the equations of motion. Nor does it affect $G_2(x_1 \neq x_2)$.

Solving Wave Equation II (k -space)

Since the previous approach isn’t always available (for example if there is a black hole in the spacetime), let’s redo the calculation in k -space.

Return to wave equation

$$0 = [z^{D+1} \partial_z (z^{-D+1} \partial_z) - m^2 L^2 - z^2 k^2] f_k(z)$$

with $k^2 = -\omega^2 + \mathbf{k}^2 > 0$. The solution is

$$f_k(z) = A_K z^{\frac{D}{2}} K_\nu(kz) + A_I z^{\frac{D}{2}} I_\nu(kz),$$

with $\nu = \sqrt{(D/2)^2 + m^2 L^2} = \Delta_+ - D/2$. Assume $k \in \mathbb{R}$ (real time issues later). As $z \rightarrow \infty$: $K_\nu \sim e^{-kz}$ and $I_\nu \sim e^{kz}$. The latter is not okay, so $A_I = 0$.

At boundary:

$$K_\nu(n) \sim n^{-\nu} (a_0 + a_1 n^2 + a_2 n^4 + \dots) + \begin{cases} n^\nu (b_0 + b_1 n^2 + b_2 n^4 + \dots), & \nu \notin \mathbb{R} \\ n^\nu \ln n (b_0 + b_1 n^2 + b_2 n^4 + \dots), & \nu \in \mathbb{R} \end{cases}$$

Hence

$$f_k(z) = A_K z^{D/2} K_\nu(kz) \sim z^{\frac{D}{2} \pm \nu} = z^{\Delta_\pm}, \quad \text{as } z \rightarrow 0.$$