

Problem Set 2 Solution

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Bulmer Exercise 2.1

When you are calculating conditional proportion (or probability), your sample space differs from that of marginal or joint proportions. For example, when it is conditioned by 6 on white die, only the sample size of 6 on white die (here, therefore, 3932) is a sample set to be considered. That is, the denominator of a probability completely changes. Based on this fact, calculating conditional probability yields:

(a) 3 on red die given 6 on white die = $629/3932 = .16$

(b) 5 on white die given 4 on red die = $509/2916 = .175$

Both of (a) and (b) are different from marginal probabilities of .159 and .182 respectively. Although not perfectly same, marginal probabilities and conditional probabilities are very close. Since this is a rolling a die game, two events of throwing of a white die and a red die should be independent. Moreover, if two events are independent of each other, we expect that a marginal probability is same as a conditional probability. Why? Recollect that

$$\Pr(A|B) = \frac{\Pr(A \& B)}{\Pr(B)}$$

If A and B are independent events, joint probability of A and B is simply a production of A and B . Therefore,

$$\begin{aligned} \Pr(A|B) &= \frac{\Pr(A \& B)}{\Pr(B)} = \frac{\Pr(A) \Pr(B)}{\Pr(B)} \\ &= \Pr(A) \end{aligned}$$

Thus, when two events are independent of each other, a conditional probability is the same as a marginal probability.

Bulmer Exercise 2.5

The sample set here is all dates in a year. Therefore, there are 365 days to choose in a sample set. (we ignore a leap year with 366 days here.) First person has all 365 days to choose as his birthday. His probability of having a specific birthday will be always $\frac{365}{365} = 1$, since he can freely choose any day.

(1) When no one shares the same birthday, the choice set left for second person only contains 364 days. Therefore, his probability of choosing any

day except for first person's birthday is $\frac{364}{365}$. In the same way, the probability that third person's birthday is not same as first persons' nor second persons' is $\frac{363}{365}$. The probability of three events are happening together is a production of each of three probabilities, $\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} = .9918$

(2) When two people share the same birthday, first person choose any day he wants, so his probability is 1 again. If second person's birthday is the same as first one, he is only able to choose the same date for his birthday as first person's. Therefore, his sample set contains only one date, which is first person's birthday. Therefore, his probability is $\frac{1}{365}$. Lastly, third person is left with 364 days, which are different from first and second person's birthday, to choose. Therefore, his probability is $\frac{364}{365}$. Now, the joint probability is $\frac{365}{365} \times \frac{1}{365} \times \frac{364}{365} = .0027$. We should consider there are three birthday-sharing pairs out of three people. So, multiplying the joint probability by 3 yields $\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times 3 = .0082$.

(3) The probability of all three sharing the same birthday is $\frac{365}{365} \times \frac{1}{365} \times \frac{1}{365}$, since both second and third persons's sample set only has one date which is first person's birthday.

Bulmer Problem 2.3

Pairwise independent if

$$P(E_i \& E_j) = P(E_i) \times P(E_j) \text{ for all } i \neq j.$$

And, *mutually independent if*

$$P(E_1 \& E_2 \& \dots \& E_n) = P(E_1) \times P(E_2) \dots \times P(E_n) \text{ for all } 1 \neq 2 \dots n.$$

The probability of first die and second die to have an odd number is $\frac{1}{2}$ for both. Also, the probability of having an odd sum of two dice is $\frac{1}{2}$. (p.20, Table 4)

Apparently, having an odd number on first die is an independent event of having an odd number on second die. The table for the combinations of two dice follows.

Table of Die Combination

First Die	Second Die	Sum of Two Dice
odd	odd	even
odd	even	odd
even	odd	odd
even	even	even

As the table shows, the probability of having an odd number on first die **and** an odd number on second die is $\frac{1}{4}$. (Note that there is only one case

that satisfies this out of four cases.) And the probability of an odd number on first die and an odd sum of two dice is also $\frac{1}{4}$. Lastly, the joint probability of an odd number on second die and an odd sum of two dice is $\frac{1}{4}$.

If pairwise independence exists, joint probabilities should be the same as the product of events' probabilities.

$$\begin{aligned}\Pr(E_1 \& E_2) &= \frac{1}{4} = \Pr(E_1) \times \Pr(E_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \\ \Pr(E_2 \& E_3) &= \frac{1}{4} = \Pr(E_2) \times \Pr(E_3) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \\ \Pr(E_1 \& E_3) &= \frac{1}{4} = \Pr(E_1) \times \Pr(E_3) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\end{aligned}$$

Therefore, three events are pairwise independent.

In the same way, mutual independence should satisfy:

$$\Pr(E_1 \& E_2 \& E_3) = 0 = \Pr(E_1) \times \Pr(E_2) \times \Pr(E_3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

As shown above, the joint probability of E_1 , E_2 and E_3 , which is 0, is not the same as the product of three events' probability, $\frac{1}{8}$. Thus, they are not mutually independent. And this suggests that pairwise independence does not guarantee mutual independence.

Bulmer Exercise 3.1 Please see attached graphs.

Bulmer Exercise 3.4 In order to solve this problem, I would figure out marginal probability first, then conditional probability and joint probability. The key point of the question is dependence of two variables. Unlike previous questions of independent cases, X and Y here are definitely dependent of each other, since Y is determined by X . Since a probability for any X is equally distributed to $-1, 0$ and 1 ,

$$\Pr(X = -1) = \Pr(X = 0) = \Pr(X = 1) = \frac{1}{3}$$

Y can have two outcomes of 0 and 1 . The marginal probability of Y is

$$\Pr(Y = 0) = \frac{1}{3} \quad \Pr(Y = 1) = \frac{2}{3}$$

Conditional probability of X s given Y s are:

$$\begin{aligned}\Pr(X = -1|Y = 0) &= 0 & \Pr(X = 0|Y = 0) &= 1 & \Pr(X = 1|Y = 0) &= 0 \\ \Pr(X = -1|Y = 1) &= \frac{1}{2} & \Pr(X = 0|Y = 1) &= 0 & \Pr(X = 1|Y = 1) &= \frac{1}{2}\end{aligned}$$

The joint probability can be calculated using conditional probability and marginal probability. Recall that event A and B 's joint probability is:

$$\Pr(A \& B) = \Pr(A|B) \times \Pr(B) = \Pr(B|A) \times \Pr(A)$$

Inserting what we know already into formula above yields:

$$\Pr(X = -1 \& Y = 0) = \Pr(X = -1|Y = 0) \times \Pr(Y = 0) = 0 \times \frac{1}{3} = 0$$

$$\Pr(X = 0 \& Y = 0) = \Pr(X = 0|Y = 0) \times \Pr(Y = 0) = 1 \times \frac{1}{3} = \frac{1}{3}$$

$$\Pr(X = 1 \& Y = 0) = \Pr(X = 1|Y = 0) \times \Pr(Y = 0) = 0 \times \frac{1}{3} = 0$$

$$\Pr(X = -1 \& Y = 1) = \Pr(X = -1|Y = 1) \times \Pr(Y = 1) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

$$\Pr(X = 0 \& Y = 1) = \Pr(X = 0|Y = 1) \times \Pr(Y = 1) = 0 \times \frac{2}{3} = 0$$

$$\Pr(X = 1 \& Y = 1) = \Pr(X = 1|Y = 1) \times \Pr(Y = 1) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$