**Problem Set 4 Solution**

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**Bulmer Problem 3.1** Since $x$ is uniformly distributed, you can have the distribution function of $x$. (Uniform distribution between [a,b] is $\frac{1}{b-a}$.)

$$X \sim U[0,1] \text{ then, } f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Then, what is $G(y)$ and $g(y)$?

(a) $y = x^2$

According to the rule demonstrated in the book,

$$\begin{aligned} G(y) &= \Pr[Y \le y] = \Pr[x^2 \le y] \\ &= \Pr[-\sqrt{y} \le x \le \sqrt{y}] \\ &= \Pr[-\sqrt{y} \le x \le 0] + \Pr[0 \le x \le \sqrt{y}] \\ &= \Pr[0 \le x \le \sqrt{y}] \ \leftarrow \ \text{because } x \text{ is between 0 and 1} \\ &= \int_0^{\sqrt{y}} f(x)\ dx = F(\sqrt{y}) - F(0) \\ &= [x]_0^{\sqrt{y}} = \sqrt{y} \end{aligned}$$

$$g(y) = \frac{dG(y)}{dy} = \frac{d\sqrt{y}}{dy} = \frac{1}{2} \cdot y^{-\frac{1}{2}}$$

(b) $y = \sqrt{x}$

$$\begin{aligned} G(y) &= \Pr[Y \le y] = \Pr[\sqrt{x} \le y] \\ &= \Pr[x \le y^2] \\ &= \int_0^{y^2} f(x)\ dx = F(y^2) - F(0) \\ &= [x]_0^{y^2} = y^2 \end{aligned}$$

$$g(y) = \frac{dG(y)}{dy} = \frac{dy^2}{dy} = 2y$$

**Bulmer Problem 3.2**

(a) First, the proof when $b$ is a negative number follows.

$$G(y) \quad = \quad Pr[Y \leq y] = Pr[a + bX \leq y] = Pr[bX \leq y - a] = Pr\left[X \geq \frac{y-a}{b}\right]$$

Since the total probability of any distribution is 1, $Pr\left[X \geq \frac{y-a}{b}\right]$ will be the same as $1 - Pr\left[X \leq \frac{y-a}{b}\right]$. (note: you can also consider this as same as $Pr\left[X \leq -\frac{y-a}{b}\right]$, but it only woks when the symmetry of a distribution is assumed.) Therefore,

$$\begin{aligned} G(y) &= 1 - Pr\left[X \leq \frac{y-a}{b}\right] = 1 - F\left(\frac{y-a}{b}\right) \\ g(y) &= \frac{dG(y)}{dy} = -\frac{1}{b} f\left(\frac{y-a}{b}\right) \end{aligned}$$

by Chain rule.

(b) Now, the application. When $Y = -log_e x$, find $g(y)$. First, $X$ is assumed to be uniformly distributed between 0 and 1. From this assumption, we can induce $X$'s probability density function of $f(x)$. It is

$$f(x) \quad \begin{cases} 1 & \text{if } 1 \leq x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$f(x)$ is always a constant of 1, since the probability of $f(x)$ is evenly 1 at any point on $x$-axis. Using the procedure in (a),

$$\begin{aligned} G(y) &= Pr\left[Y \leq y\right] = Pr\left[-log_e x \leq y\right] = Pr\left[log_e x \geq y\right] \\ &= 1 - Pr\left[log_e x \leq -y\right] \ = 1 - Pr\left[x \leq e^{-y}\right] \\ &= 1 - F\left(e^{-y}\right) \\ g(y) &= \frac{dG(y)}{dy} = -f(e^{-y})(-1)e^{-y} = e^{-y} \end{aligned}$$

**Bulmer Problem 3.3**

This is a similar question as the first one, but with different $X$ distribution. Now, $X$ is uniformly distributed between $-1$ and 2, which makes the probability density function of $X$ is

$$f(x) \begin{cases} \frac{1}{3} & \text{if } -1 \leq x \leq 2 \\ \\ 0 & \text{otherwise} \end{cases}$$

And $Y = X^2$. The transformation process yields

$$G(y) = Pr\left[Y \leq y\right] = Pr\left[X^2 \leq y\right] = Pr\left[-\sqrt{y} \leq x \leq \sqrt{y}\right]$$

Here, we have to consider the range of possible $x$s. Differing from the Problem 3.1, $x$ has the range of $[-1, 2]$. Between $[-1, 1]$, $Y$ is non-monotonous function and then monotonously increasing between $[1, 2]$. The distribution function should be different for these two different ranges.

First, I consider when $x$ is between $[-1, 1]$. The transformation is:

$$\begin{aligned} G(y) &= Pr\left[-\sqrt{y} \leq x \leq \sqrt{y}\right] = \int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx \\ &= F(\sqrt{y}) - F(-\sqrt{y}) \\ g(y) &= \frac{dG(y)}{dy} = f(\sqrt{y})\frac{1}{2}y^{-\frac{1}{2}} - f(\sqrt{y})\left(-\frac{1}{2}\right)y^{-\frac{1}{2}} \\ &= \frac{1}{3}y^{\frac{1}{2}} \end{aligned}$$

This works if $0 < y \leq 1$ since the function is not defined if $y$ is 0.

Now, consider the case when $x$ is between $[1, 2]$. As noted, it is increasingly monotonous with a lower boundary. Therefore,

$$
\begin{aligned}
G(y) &= Pr\left[1 \leq x \leq \sqrt{y}\right] = \int_1^{\sqrt{y}} f(x)dx = F(\sqrt{y}) - F(1) \\
g(y) &= \frac{dG(y)}{dy} = f(\sqrt{y})\frac{1}{2}y^{\frac{1}{2}} \\
&= \frac{1}{6}y^{\frac{1}{2}}
\end{aligned}
$$

This is when $1 < y \leq 4$.

**Experimental Question**

For the Stata result, please see attached.
Theoretical mean can be obtained by using the following formula.

$$
\begin{aligned}
E(x) &= \int_0^1 xf(x)\ dx = \int_0^1 x \cdot 1\ dx = \left[\frac{1}{2}x^2\right]_0^1 \\
&= \frac{1}{2} \\
Var(x) &= \int_0^1 (x - E(x))^2\ dx = \int_0^1 x^2 - x + \tfrac{1}{4}\ dx \\
&= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x\right]_0^1 = \frac{1}{12}
\end{aligned}
$$

Now, compare these with Stata result.