

Problem Set 4 Solution

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Bulmer Exercise 5.1

If you directly calculate the mean and the variance, you need to use the formula of

$$\begin{aligned} E(X) &= \sum xf(x) \\ Var(x) &= \sum (x - \mu)^2 f(x) \end{aligned}$$

since it is a discrete case. Given the table with the number of events and the probabilities of an event happening, it will yield the mean of 10.5 and variance of 8.75.

However, we can more easily calculate the mean and the variance of sum of three dice. Recall that :

$$\begin{aligned} E(X + Y) &= E(X) + E(Y) \\ Var(X + Y) &= Var(X) + Var(Y) + 2Cov(X, Y) \end{aligned}$$

Since the expected value of one die is $E(X_1) = 3.5$, $E(X_1 + X_2 + X_3) = 3.5 + 3.5 + 3.5 = 10.5$. Throwing a die is not influenced by throwing another die. That is, each event of throwing a die is independent of each other, which makes covariance zero. Thus, $Var(X_1 + Var_2 + Var_3) = Var(X_1) + Var(X_2) + Var(X_3) = 8.75$. Compared with direct results, these are correct, too.

Bulmer Exccercise 5.3

First, let's try to prove that $Cov(X, -Y) = -Cov(X, Y)$. Inserting X and $-Y$ into a covariance formula yields

$$\begin{aligned} Cov(X, -Y) &= E[(X - E(X))(-Y - E(-Y))] = -E[(X - E(X))(Y - E(Y))] \\ &= -Cov(X, Y) \end{aligned}$$

From this result, we can also prove that $Var(X - Y) = Var(X) + Var(Y) - 2Cov(X, Y)$.

$$\begin{aligned}
\text{Var}(X - Y) &= E [((X - Y) - E(X - Y))^2] = E [(X - Y - E(X) + E(Y))^2] \\
&= E [((X - E(X)) - (Y - E(Y)))^2] \\
&= E [(X - E(X))^2 + (Y - E(Y))^2 - 2(X - E(X))(Y - E(Y))] \\
&= E [(X - E(X))^2] + E [(Y - E(Y))^2] - 2E [(X - E(X))(Y - E(Y))] \\
&= \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y)
\end{aligned}$$

Can it work more simply? Yes. Let's consider that $\text{Var}(X - Y) = \text{Var}(X + (-1)Y)$. Then,

$$\begin{aligned}
\text{Var}(X + (-1)Y) &= \text{Var}(X) + \text{Var}((-1)Y) + 2\text{Cov}(X, -Y) \\
\text{Var}((-1)Y) &= \text{Var}(Y) \quad \text{since } \text{Var}((-1)Y) = E [(-Y - E(-Y))^2] \\
&= E [(-1)^2(Y - E(Y))^2] = E [(Y - E(Y))^2] = \text{Var}(Y)
\end{aligned}$$

And we know from the above proof that $\text{Cov}(X, -Y) = -\text{Cov}(X, Y)$. Therefore, $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y)$.

Now, let's apply this to the question. Let a brother 1 is X_1 and a brother 2 is X_2 . Since both brothers are coming from the same population, their expected values are as same as 5ft. 8in which is the mean of men's height. Also, standard deviations are 2.5 in for both brothers.

Using formulas we obtained from above, the expected value of a sum of brothers will be $E(X_1 + X_2)$ and it will simply be the sum of two expected values of $E(X_1)$ and $E(X_2)$, which is 11.4 in. In the same way, the expected value of a difference of brothers will be $E(X_1 - X_2) = E(X_1) - E(X_2) = 0$.

Variance will be a little bit tricky. In order to figure out the variance of two variables's sum or difference, we need to know the covariance of those two variables. We are not given covariance in the question, but we have correlation coefficient given. Correlation coefficient is

$$\rho = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\text{Var}(X_1)}\sqrt{\text{Var}(X_2)}}$$

Inserting given correlation coefficient of .5 into the above equation (and we know what standard deviations for X_1 and X_2 are. They are both 2.5 in.)

will yield $Cov(X_1, X_2) = 3.125$. Now, plug all information we have into $Var(X_1 + X_2)$ and $Var(X_1 - X_2)$ equations, then you will get 6.25 sq. in and 18.75 sq. in respectively.

Snedecor & Cochran 4.3.1

(i) This is the case when you have $\mu = 0$ and $\sigma = 2$. If you had $\sigma = 1$, you have a standard normal distribution. In standard normal Z table, the height at the mean is .3989 (at the end of the book, you will find the value at $Z = 0$). Since your standard deviation is 2, you have divide this number by 2, which is $\frac{.3989}{2} = .1994$. You can imagine that a wider normal curve when you have higher variance.

(ii) The highest value of a curve's height is when a curve is at its mean. Therefore, one-tenth of the highest height is .03989, and the Z value at its height is about 2.15 as shown in the book. Therefore, at $X = \mu + 2.15\sigma$, the height of the curve is .03989.

Snedecor & Cochran 4.3.2

It is a simple algebra question. If you look at the Z -table, from 0 to 1.76, the area under the curve is .4608. Since it is a symmetric curve, you simply double the area, which yields .9216. In percentage term, therefore, it is 92.16 %.

Snedecor & Cochran 4.3.3

Again, from 0 to -1.1, the area is .3643. (This can be obtained by looking at from 0 to 1.1 because the curve is symmetric.) And, 0 to .8 is .2881. Simple addition of .3643 and .2881 is .6524, which is 65.24 %.

Snedecor & Cochran 4.3.4

Now, we have an area and of 13.59%, which is .1359. Look at the table again, and find the value under $Z = 1$ and $Z = 2$. These are .3413 and .4772 respectively. The area between them is, therefore, $.4772 - .3413 = .1359$, which is 13.59%.

Snedecor & Cochran 4.3.5

If half the population lies in the interval from $\mu - .6745\sigma$ to $\mu + .6745\sigma$, the area between this range will be .5. In the table, we do not have an exact number for .6745 in the table, but we can approximately calculate it as an middle point between .67 and .68. This is $(.2486 + .2517)/2 = .25015$. Since the table only shows a positive Z s, we expect that the area from 0 to .6745 is 25% not 50%. Therefore, a quarter of population lies in 0 and .6745, half the population lies in -.6745 and .6745.

Snedecor & Cochran 4.6.1

(i) Using the formula that standard error of $\hat{\sigma} = \frac{\sigma}{\sqrt{n}}$, $\frac{2.6}{\sqrt{25}} = .52$.

(ii) $\frac{2.6}{\sqrt{100}} = .26$.

Snedecor & Cochran 4.6.4

$$\begin{aligned}\mu &= \sum_{i=1}^6 xP(x) = \sum_{i=1}^6 x\frac{1}{6} = 3.5 \\ \sigma^2 &= \sum_{i=1}^6 (x - 3.5)^2 P(x) = \sum_{i=1}^6 (-3.5)^2 x\frac{1}{6} = 2.916 \\ \sigma &= \sqrt{\sigma^2} = 1.71\end{aligned}$$

Snedecor & Cochran 4.6.5

$$\begin{aligned}\mu &= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1 \\ \sigma^2 &= (0 - 1)^2 \frac{1}{4} + (1 - 1)^2 \cdot \frac{1}{2} + (2 - 1)^2 \cdot \frac{1}{4} = \frac{1}{2} \\ \sigma &= \frac{1}{\sqrt{2}}\end{aligned}$$