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Martingale Convergence Theorem

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1 Martingale Convergence Theorem

Theorem 1. (Doob) *Suppose X_n is a super-martingale which satisfies*

$$\sup_n \mathbb{E}[|X_n|] < \infty$$

Then, almost surely $X_\infty = \lim_n X_n$ exists and is finite in expectation. That is, define $X_\infty = \limsup X_n$. Then $X_n \to X_\infty$ a.s. and $\mathbb{E}[|X_\infty|] < \infty$.

Proof. The proof relies "Doob's Upcrossing Lemma". For that consider

$$\begin{aligned}\Lambda &\triangleq \{\omega : X_n(\omega) \text{ does not converge to a limit in } \mathbb{R}\}\\ &= \{\omega : \liminf_n X_n(\omega) < \limsup_n X_n(\omega)\}\\ &= \cup_{a<b:a,b\in\mathbb{Q}}\{\omega : \liminf_n X_n(\omega) < a < b < \limsup_n X_n(\omega)\},\end{aligned} \tag{1}$$

where $\mathbb{Q}$ is the set of rational values. Let, $U_N[a,b](\omega) =$ largest k such that it satisfies the following: there exists

$$0 \le s_1 < t_1 < ... < s_k < t_k \le N$$

such that

$$X_{s_i}(\omega) < a < b < X_{t_i}(\omega), \qquad 1 \le i \le k.$$

That is, $U_N[a,b]$ is the number of up-crossings of $[a,b]$ up to N. Clearly, $U_N[a,b](\omega)$ is non-decreasing in N. Let $U_\infty[a,b](\omega) = \lim_{N\to\infty} U_N[a,b](\omega)$. Then (1) can be re-written as

$$\begin{aligned} \Lambda &= \cup_{a<b:a,b\in\mathbb{Q}}\{\omega : U_\infty[a,b](\omega) = \infty\} \\ &= \cup_{a<b:a,b\in\mathbb{Q}}\Lambda_{a,b}. \end{aligned} \tag{2}$$

Doob's upcrossing lemma proves that $\mathbb{P}(\Lambda_{a,b}) = 0$ for every $a < b$. Then we have from (2) that $\mathbb{P}(\Lambda) = 0$. Thus, $X_n(\omega)$ converges in $[-\infty, \infty]$ a.s. That is,

$$X_\infty = \lim_n X_n \text{ exists a.s.}$$

Now,

$$\begin{aligned} \mathbb{E}[|X_\infty|] &= \mathbb{E}[\liminf_n |X_n|] \\ &\le \liminf_n \mathbb{E}[|X_n|] \\ &\le \sup_n \mathbb{E}[|X_n|] < \infty, \end{aligned}$$

where we have used Fatou's Lemma in the first inequality. Thus, X_∞ is in $\mathbb{L}_1$. In particular, X_∞ is finite a.s. This completes the proof of Theorem 5 (with Doob's upcrossing Lemma and its application $P(\Lambda_{a,b}) = 0$ remaining to be proved.) □

Lemma 1. (Doob's Upcrossing) *Let X_n be a super-MG. Let $U_N[a,b]$ be the number of upcrossing of $[a,b]$ until time N with $a < b$. Then,*

$$(b-a)\mathbb{E}[U_N[a,b]] \le \mathbb{E}[(X_N - a)^-]$$

where

$$(X_N - a)^- = \begin{cases} a - X_N, & \textit{if } X_N \le a \\ 0, & \textit{otw.} \end{cases}$$

Proof. Define a predictable sequence C_n as follows.

$$C_1(\omega) = \begin{cases} 1, & \text{if } X_0(\omega) < a \\ 0, & \text{otw.} \end{cases}$$

Inductively,

$$C_n(\omega) = \begin{cases} 1, & \text{if } C_{n-1}(\omega) = 1 \text{ and } X_{n-1}(\omega) \le b \\ 1, & \text{if } C_{n-1}(\omega) = 0 \text{ and } X_{n-1}(\omega) < a \\ 0, & \text{otw.} \end{cases}$$

By definition, C_n is predictable. The sequence C_n has the following property. If $X_0 < a$ then $C_1 = 1$. Then the sequence C_n remains equal to 1 until the first time X_n exceeds b. It then remains zero until the first time it becomes smaller than a at which point it switches back to 1, etc. If instead $X_0 > a$, then $C_1 = 0$ and it remains zero until the first time X_n becomes smaller than a, at which point C_n switches to 1 and then continues as above. Consider

$$Y_n = (C \cdot X)_n = \sum_{1 \le k \le n} C_k (X_k - X_{k-1})$$

We claim that

$$Y_N(\omega) \ge (b-a) U_N[a,b] - (X_N(\omega) - a)^-.$$

Let $U_N[a,b] = k$. Then there is $0 \le s_1 < t_1 < \cdots < s_k < t_k \le N$ such that $X_{s_i}(\omega) < a < b < X_{t_i}(\omega)$, $i = 1, \cdots, k$. By definition, $C_{s_i+1} = 1$ for all $i \ge 1$. Further, $C_t(\omega) = 1$ for $s_i + 1 \le t \le l_i \le t_i$ where $l_i \le t_i$ is the smallest time $t \ge s_i$ such that $X_t(\omega) > b$. Without the loss of generality, assume that $s_1 = \min\{n : X_n < a\}$. Let, $s_{k+1} = \min\{n > t_k : X_n(\omega) < a\}$. Then,

$$\begin{aligned} Y_N(\omega) &= \sum_{j \le N} C_j(\omega)(X_j(\omega) - X_{j-1}(\omega)) \\ &= \sum_{1 \le i \le k} [\sum_{s_i \le t \le l_i} C_{t+1}(\omega)(X_{t+1}(\omega) - X_t(\omega))] \\ &\quad + \sum_{t \ge s_{k+1}} C_{t+1}(\omega)(X_{t+1}(\omega) - X_t(\omega)) \text{ (Because otherwise } C_t(\omega) = 0.) \\ &= \sum_{1 \le i \le k} (X_{l_i}(\omega) - X_{s_i}(\omega)) + X_N(\omega) - X_{s_{k+1}}(\omega), \end{aligned}$$

where the term $X_N(\omega) - X_{s_{k+1}}(\omega)$ is defined to be zero if $s_{k+1} > N$. Now, $X_{l_i}(\omega) - X_{s_i}(\omega) \ge b - a$. Now if $X_N(\omega) \ge X_{s_{k+1}}$ then

$$X_N(\omega) - X_{s_{k+1}} \ge 0.$$

Otherwise

$$|X_N(\omega) - X_{s_{k+1}(\omega)}| \le |X_N(\omega) - a|.$$

Therefore we have

$$Y_N(\omega) \geq U_N[a,b](b-a) - (X_N(\omega) - a)^-$$

as claimed.

Now, as we have established earlier, $Y_n = (C \cdot X)_n$ is super-MG since $C_n \geq 0$ is predictable. That is,

$$\mathbb{E}[Y_N] \leq \mathbb{E}[Y_0] = 0$$

By claim,

$$(b-a)\mathbb{E}[U_N[a,b]] \leq \mathbb{E}[(X_N - a)^-]$$

This completes the proof of Doob's Lemma. □

Next, we wish to use this to prove $\mathbb{P}(\Lambda_{a,b}) = 0$.

Lemma 2. *For any $a < b$, $\mathbb{P}(\Lambda_{a,b}) = 0$.*

Proof. By definition $\Lambda_{a,b} = \{\omega : U_\infty[a,b] = \infty\}$. Now by Doob's Lemma

$$\begin{aligned}(b-a)\mathbb{E}[U_N[a,b]] &\leq \mathbb{E}[(X_N - a)^-] \\ &\leq \sup_n \mathbb{E}[|X_n|] + |a| \\ &< \infty\end{aligned}$$

Now, $U_N[a,b] \nearrow U_\infty[a,b]$. Hence by the Monotone Convergence Theorem, $\mathbb{E}[U_N[a,b]] \nearrow \mathbb{E}[U_\infty[a,b]]$. That is, $\mathbb{E}[U_\infty[a,b]] < \infty$. Hence, $\mathbb{P}(U_\infty[a,b] = \infty) = 0$. □

2 Doob's Inequality

Theorem 2. *Let X_n be a sub-MG and let $X_n^* = \max_{0 \leq m \leq n} X_m^+$. Given $\lambda > 0$, let $A = \{X_n^* \geq \lambda\}$. Then,*

$$\lambda\mathbb{P}(A) \leq \mathbb{E}[X_n \mathbf{1}(A)] \leq \mathbb{E}[X_n^+]$$

Proof. Define stopping time

$$N = \min\{m : X_m^* \geq \lambda \text{ or } m = n\}$$

Thus, $\mathbb{P}(N \leq n) = 1$. Now, by the Optional Stopping Theorem we have that $X_{N \wedge n}$ is a sub-MG. But $X_{N \wedge n} = X_N$. Thus

$$\mathbb{E}[X_N] \leq \mathbb{E}[X_n] \tag{3}$$

We have

$$\begin{aligned}\mathbb{E}[X_N] &= \mathbb{E}[X_N \mathbf{1}(A)] + \mathbb{E}[X_N \mathbf{1}(A^c)] \\ &= \mathbb{E}[X_N \mathbf{1}(A)] + \mathbb{E}[X_n \mathbf{1}(A^c)] \end{aligned} \tag{4}$$

Similarly,

$$\mathbb{E}[X_n] = \mathbb{E}[X_n \mathbf{1}(A)] + \mathbb{E}[X_n \mathbf{1}(A^c)] \tag{5}$$

From (3)∼ (5), we have

$$\mathbb{E}[X_N \mathbf{1}(A)] \leq \mathbb{E}[X_n \mathbf{1}(A)] \tag{6}$$

But

$$\lambda \mathbb{P}(A) \leq \mathbb{E}[X_N \mathbf{1}(A)] \tag{7}$$

From (6) and (7),

$$\begin{aligned}\lambda \mathbb{P}(A) &\leq \mathbb{E}[X_n \mathbf{1}(A)] \\ &\leq \mathbb{E}[X_n^+ \mathbf{1}(A)] \\ &\leq \mathbb{E}[X_n^+]. \end{aligned} \tag{8}$$

□

Suppose, X_n is non-negative sub-MG. Then,

$$\mathbb{P}(\max_{0 \leq k \leq n} X_k \geq \lambda) \leq \frac{1}{\lambda} \mathbb{E}[X_n]$$

If it were MG, then we also obtain

$$\mathbb{P}(\max_{0 \leq k \leq n} X_k \geq \lambda) \leq \frac{1}{\lambda} \mathbb{E}[X_n] = \frac{1}{\lambda} \mathbb{E}[X_0]$$

3 L^p maximal inequality and L^p convergence

Theorem 3. *Let, X_n be a sub-MG. Suppose $\mathbb{E}[(X_n^+)^p] < \infty$ for some $p > 1$. Then,*

$$\mathbb{E}[(\max_{0 \leq k \leq n} X_k^+)^p]^{\frac{1}{p}} \leq q \mathbb{E}[(X_n^+)^p]^{\frac{1}{p}}$$

where $\frac{1}{q} + \frac{1}{p} = 1$. In particular, if X_n is a MG then $|X_n|$ is a sub-MG and hence

$$\mathbb{E}[(\max_{0 \leq k \leq n} |X_k|)^p]^{\frac{1}{p}} \leq q \mathbb{E}[|X_n|^p]^{\frac{1}{p}}$$

Before we state the proof, an important application is

Theorem 4. *If X_n is a martingale with $\sup_n \mathbb{E}[|X_n|^p] < \infty$ where $p > 1$, then $X_n \to X$ a.s. and in L^p, where $X = \limsup_n X_n$.*

We will first prove Theorem 4 and then Theorem 3.

Proof. (Theorem 4) Since $\sup_n \mathbb{E}[|X_n|^p] < \infty$, $p > 1$, by MG-convergence theorem, we have that

$$X_n \to X, \text{ a.s., where } X = \limsup_n X_n.$$

For L^p convergence, we will use L^p-inequality of Theorem 3. That is,

$$\mathbb{E}[(\sup_{0\le m\le n} |X_m|)^p] \le q^p \mathbb{E}[|X_n|^p]$$

Now, $\sup_{0\le m\le n}|X_m| \nearrow \sup_{0\le m}|X_m|$. Therefore, by the Monotone Convergence Theorem we obtain that

$$\mathbb{E}[\sup_{0\le m}|X_m|^p] \le q^p \sup_n \mathbb{E}[|X_n|^p] < \infty$$

Thus, $\sup_{0\le m}|X_m| \in L^p$. Now,

$$|X_n - X| \le 2 \sup_{0\le m}|X_m|$$

Therefore, by the Dominated Convergence Theorem $\mathbb{E}[|X_n - X|^p] \to 0$. □

Proof. (Theorem 3) We will use truncation of X_n^* to prove the result. Let M be the truncation parameter: $X_n^{*,M} = \min(X_n^*, M)$. Now, consider the following:

$$\begin{aligned}\mathbb{E}[(X_n^{*,M})^p] &= \int_0^\infty p\lambda^{p-1}\mathbb{P}(X_n^{*,M} \ge \lambda)d\lambda \\ &\le \int_0^\infty p\lambda^{p-1}[\frac{1}{\lambda}\mathbb{E}[X_n^+ \mathbf{1}(X_n^{*,M} \ge \lambda)]]d\lambda\end{aligned}$$

The above inequality follows from

$$\mathbb{P}(X_n^{*,M} \ge \lambda) = \begin{cases} 0, & \text{if } M < \lambda \\ \mathbb{P}(X_n^* \ge \lambda), & \text{if } M \ge \lambda \end{cases}$$

and Theorem 2. By an application of Fubini for non-negative integrands, we have

$$
\begin{aligned}
& p\mathbb{E}[X_n^+ \int_0^{X_n^{*,M}} \lambda^{p-2} d\lambda] \\
& = \frac{p}{p-1}\mathbb{E}[X_n^+ (X_n^{*,M})^{p-1}] \\
& \leq \frac{p}{p-1}\mathbb{E}[(X_n^+)^p]^{\frac{1}{p}}\mathbb{E}[(X_n^{*,M})^{(p-1)q}]^{\frac{1}{q}}, \text{ by Holder's inequality.} \qquad (9)
\end{aligned}
$$

Here, $\frac{1}{q} = 1 - \frac{1}{p} \Rightarrow q(p-1) = p$. Thus, we can simplify (9)

$$= q\mathbb{E}[(X_n^+)^p]^{\frac{1}{p}}\mathbb{E}[(X_n^{*,M})^p]^{\frac{1}{q}}$$

Thus,

$$||X_n^{*,M}||_p^p \leq q||X_n^+||_p||X_n^{*,M}||_p^{\frac{p}{q}}$$

That is,

$$||X_n^{*,M}||_p^{p(1-\frac{1}{q})} \leq q||X_n^+||_p$$

Hence, $||X_n^{*,M}||_p \leq q||X_n^+||_p$. □

4 Backward Martingale

Let $\mathcal{F}_n$ be increasing sequence of σ-algebra, $n \leq 0$, such that $\cdots \subset \mathcal{F}_{-3} \subset \mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset F_0$. Let X_n be $\mathcal{F}_n$ adapted, $n \leq 0$, and

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n, \ n < 0$$

Then X_n is called backward MG.

Theorem 5. *Let X_n be backward MG. Then*

$$\lim_{n \to -\infty} X_n = X_{-\infty} \text{ exists a.s. and in } L^1.$$

Compare with standard MG convergence results:
(a): We need $\sup E[|X_n|] < \infty$, or non-negative MG in Doob's convergence theorem, which gives a.s. convergence not L^1.
(b): For L^1, we need UI. And, it is necessary because if $X_n \to X_{infty}$ a.s. and L^1 then there exists $X \in F_\infty$ s.t. $X_n = \mathbb{E}[X|F_n]$; and hence X_n is UI.

Proof of Theorem 5. Recall Doob's convergence theorem's proof. Let

$$\begin{aligned}\Lambda :&= \{\omega : X_n(\omega) \text{ does not converge to a limit in } [-\infty, \infty]\}\\ &= \{\omega : \liminf_n X_n(\omega) < \limsup_n X_n(\omega)\}\\ &= \cup_{a,b:a,b\in\mathcal{Q}}\{\omega : \liminf_n X_n(\omega) < a < b < \limsup_n X_n(\omega)\}\\ &= \cup_{a,b:a,b\in\mathcal{Q}}\Lambda_{a,b}\end{aligned}$$

Now, recall $U_n[a, b]$ is the number of upcrossing of $[a, b]$ in $X_n, X_{n+1}, ..., X_0$ as $n \to -\infty$. By upcrossing inequality, it follows that

$$(b - a)\mathbb{E}[U_n[a, b]] \leq \mathbb{E}[|X_0|] + |a|$$

Since $U_n[a, b] \nearrow U_\infty[a, b]$ and By monotone convergence theorem, we have

$$\mathbb{E}[U_\infty[a, b]] < \infty \Rightarrow \mathbb{P}(\Lambda_{a,b}) = 0$$

This implies X_n converges a.s.

Now, $X_n = \mathbb{E}[X_0|F_n]$. Therefore, X_n is UI. This implies $X_n \to X_{-\infty}$ in L^1. □

Theorem 6. *If $X_{-\infty} = \lim_{n\to-\infty} X_n$ and $\mathcal{F}_{-\infty} = \cap_n\mathcal{F}_n$. Then $X_{-\infty} = \mathbb{E}[X_0|\mathcal{F}_{-\infty}]$.*

Proof. Let $X_n = \mathbb{E}[X_0|\mathcal{F}_n]$. If $A \in \mathcal{F}_{-\infty} \subset \mathcal{F}_n$, then $\mathbb{E}[X_n; A] = \mathbb{E}[X_0; A]$. Now,

$$\begin{aligned}|\mathbb{E}[X_n; A] - \mathbb{E}[X_{-\infty}; A]| &= |\mathbb{E}[X_n - X_{-\infty}; A]|\\ &\leq \mathbb{E}[|X_n - X_{-\infty}|; A]\\ &\leq \mathbb{E}[|X_n - X_{-\infty}|] \to 0 \text{ as } n \to -\infty \text{ (by Theorem 5)}\end{aligned}$$

Hence, $\mathbb{E}[X_{-\infty}; A] = \mathbb{E}[X_0; A]$. Thus, $X_{-\infty} = \mathbb{E}[X_0|\mathcal{F}_{-\infty}]$. □

Theorem 7. *Let $\mathcal{F}_n \searrow \mathcal{F}_{-\infty}$, and $Y \in L^1$. Then, $\mathbb{E}[Y|\mathcal{F}_n] \to \mathbb{E}[Y|\mathcal{F}_{-\infty}]$ a.s. in L^1.*

Proof. $X_n = \mathbb{E}[Y|\mathcal{F}_n]$ is backward MG by definition. Therefore,

$$X_n \to X_{-\infty} \text{ a.s. and in } L^1.$$

By Theorem 6, $X_{-\infty} = \mathbb{E}[X_0|\mathcal{F}_{-\infty}] = \mathbb{E}[Y|\mathcal{F}_{-\infty}]$. Thus, $\mathbb{E}[Y|\mathcal{F}_n] \to \mathbb{E}[Y|\mathcal{F}_{-\infty}]$. □

5 Strong Law of Large Number

Theorem 8 (SLLN). *Let ξ be i.i.d. with $\mathbb{E}[|\xi_i|] < \infty$. Let $S_n = \xi_1 + ... + \xi_n$. Let $X_{-n} = \frac{S_n}{n}$. And, $\mathcal{F}_{-n} = \sigma(S_n, \xi_{n+1}, ...)$. Then,*

$$\begin{aligned}\mathbb{E}[X_{-n}|\mathcal{F}_{-n-1}] &= \mathbb{E}[\frac{S_n}{n}|\mathcal{F}_{-n-1}] \\ &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[\xi_i|S_{n+1}] \\ &= \mathbb{E}[\xi_1|S_{n+1}] \\ &= \frac{1}{n+1}S_{n+1} \\ &= X_{-n+1}\end{aligned}$$

Then X_{-n} is backward MG.

Proof. By Theorem 5∼ 7, we have $X_{-n} \to X_{-\infty}$ a.s. and in L^1, with $X_{-\infty} = \mathbb{E}[\xi_1|\mathcal{F}_{-\infty}]$. Now $\mathcal{F}_{-\infty}$ is in ξ (the exchangeable σ-algebra). By Hewitt-Savage (proved next) 0-1 law, ξ is trivial. That is, $\mathbb{E}[\xi_1|\mathcal{F}_{-\infty}]$ is a constant. Therefore, $\mathbb{E}[X_{-\infty}] = \mathbb{E}[\xi_1]$ is also a constant. Thus,

$$X_{-\infty} = \lim_{n\to\infty}\frac{S_n}{n} = \mathbb{E}[\xi_1]$$

□

6 Hewitt-Savage 0-1 Law

Theorem 9. *Let $X_1, ..., X_n$ be i.i.d. and ξ be the exchangeable $\sigma - algebra$:*

$$\xi_n = \{A : \pi_n A = A; \forall \pi_n \in S_n\}; \quad \xi = \cup_n \xi_n$$

If $A \in \xi$, then $\mathbb{P}(A) \in \{0, 1\}$.

Proof. The key to the proof is the following Lemma:

Lemma 3. *Let $X_1, ..., X_k$ be i.i.d. and define*

$$A_n(\phi) = \frac{1}{n_{p_k}}\sum_{(i_1,...,i_k)\in\{1,...,n\}} A_n(\phi(X_{i_1}, ..., X_{i_k}))$$

If ϕ is bounded then

$$A_n(\phi) \to \mathbb{E}[\phi(X_1, ..., X_k)] \text{ a.s.}$$

Proof. $A_n(\phi) \in \xi_n$ by definition. So

$$\begin{aligned} A_n(\phi) &= \mathbb{E}[A_n(\phi)|\xi_n] \\ &= \frac{1}{n_{p_k}} \sum_{i_1,...,i_k} \mathbb{E}[\phi(X_{i_1},...,X_{i_k})|\xi_n] \\ &= \frac{1}{n_{p_k}} \sum_{i_1,...,i_k} \mathbb{E}[\phi(X_1,...,X_k)|\xi_n] \\ &= \mathbb{E}[\phi(X_1,...,X_k)|\xi_n] \end{aligned} \tag{10}$$

Let $\mathcal{F}_{-n} = \xi_n$. Then $\mathcal{F}_{-n} \quad \mathcal{F}_{-\infty} = \xi$. Then, for $Y = \phi(X_1,...,X_k)$. $\mathbb{E}[Y|\mathcal{F}_{-n}]$ is backward MG. Therefore,

$$\mathbb{E}[Y|\mathcal{F}_{-n}] \to \mathbb{E}[Y|\mathcal{F}_{-\infty}] = \mathbb{E}[\phi(X_1,...,X_k)|\xi]$$

Thus,

$$A_n(\phi) \to \mathbb{E}[\phi(X_1,...,X_k)|\xi] \tag{11}$$

We want to show that indeed $\mathbb{E}[\phi(X_1,...,X_n)|\xi]$ is $\mathbb{E}[\phi(X_1,...,X_n)]$.

First, we show that $\mathbb{E}[\phi(X_1,...,X_n)|\xi] \in \sigma(X_{k+1},...)$ since ϕ is bounded. Then, we find that if $\mathbb{E}[X|\mathcal{G}] \in \mathcal{F}$ where X is independent of $\mathcal{F}$ then $\mathbb{E}[X|\mathcal{G}]$ is constant, equal to $\mathbb{E}[X]$. This will complete the proof of Lemma.

First step: consider $A_n(\phi)$. It has n_{p_k} terms in which there are $k(n-1)_{p_{k-1}}$ terms containing X_1. Therefore, the effect of terms containing X_1 is:

$$\begin{aligned} T_n(1) \equiv \frac{1}{n_{p_k}} \sum_{(i_1,...,i_k)} \phi(X_{i_1},...,X_{i_k}) &\leq \frac{1}{n_{p_k}} k\left((n-1)_{p_{k-1}}\right)||\phi||_\infty \\ &= \frac{(n-k)!}{n!} k \frac{(n-1)!}{(n-k)!} ||\phi||_\infty \\ &= \frac{k}{n}||\phi||_\infty \to 0 \text{ as } n \to \infty \end{aligned} \tag{12}$$

Let $A_n^{-1}(\phi) = A_n(\phi) - T_n(1)$. Then, we have $A_n^{-1}(\phi) \to \mathbb{E}[\phi(X_1,...,X_n)|\xi]$ from (11) and (12). Thus, $\mathbb{E}[\phi(\phi(X_1,...,X_n))|\xi]$ is independent on X_1. Similarly, repeating argument for $X_2,...,X_k$ we obtain that

$$\mathbb{E}[\phi(X_1,...,X_n)|\xi] \in \sigma(X_{n+1},...)$$

Second step: if $\mathbb{E}[X^2] \leq \infty$, $\mathbb{E}[X|\mathcal{G}] \in \mathcal{F}$, X is independent of $\mathcal{F}$ then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

Proof. Let $Y = \mathbb{E}[X|\mathcal{G}]$. Now $Y \in \mathcal{F}$ and X is independent on $\mathcal{F}_1$ we have that

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[Y]^2$$

, since $\mathbb{E}[Y] = \mathbb{E}[X]$. Now by definition of conditional expectation for any $Z \in \mathcal{G}$, $E[XZ] = E[YZ]$. Hence, for $Z = Y$, we have $\mathbb{E}[XY] = \mathbb{E}[Y^2]$. Thus,

$$\begin{aligned} \mathbb{E}[Y^2] = \mathbb{E}[Y]^2 &\Rightarrow Var(Y) = 0 \\ &\Rightarrow Y = \mathbb{E}[Y] \text{ a.s.} \end{aligned} \tag{13}$$

□

This completes the proof of the Lemma. □

Now completing proof of H-S law.

We have proved that $A_n(\phi) \to \mathbb{E}[\phi(X_1, ..., X_n)]$ a.s. for all bounded ϕ dependent on finitely many components.

By the first step, ξ is independent on $\mathcal{G}_k = \sigma(X_1, ..., X_k)$. This is true for all k. $\cup_k \mathcal{G}_k$ is a π-system which contains Ω. Therefore, ξ is independent of $\sigma(\cup_k \mathcal{G}_k)$ and $\xi \subset \sigma(\cup_k \mathcal{G}_k)$. Thus, for all $A \in \xi$, A is independent of itself. Hence,

$$\mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A) \Rightarrow P(A) \in \{0, 1\}.$$

□

7 De Finetti's Theorem

Theorem 10. *Given $X_1, X_2, ...$ sequence of exchangeable, that is, for any n and $\pi_n \in S_n$, $(X_1, ..., X_n) \triangleq (X_{\pi_n(1)}, ..., X_{\pi_n(n)})$, then conditional on ξ, $X_1, ..., X_n, ...$ are i.i.d.*

Proof. As in H-S's proof and Lemma, define $A_n(\phi) = \frac{1}{n_{p_k}} \sum_{(i_1,...,i_k)} \phi(X_{i_1}, ..., X_{i_k})$. Then, due to exchangeability,

$$\begin{aligned} A_n(\phi) &= \mathbb{E}[A_n(\phi)|\xi_n] = \mathbb{E}[\phi(X_1, ..., X_n)|\xi_n] \\ &\to \mathbb{E}[\phi(X_1, ..., X_n)|\xi] \text{ by backward MG convergence theorem.} \end{aligned} \tag{14}$$

Since $X_1, ...$ may not be i.i.d., ξ can be nontrivial. Therefore, the limit need not be constant. Consider a $f : \mathbb{R}^{k-1} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Let $I_{n,k}$ be set of all distinct $1 \leq i_1, ..., i_k \leq n$, then

$$\begin{aligned} & n_{p_{k-1}} A_n(f) n A_n(g) \\ &= \sum_{i \in I_{n,k-1} f(X_{i_1},...,X_{i_{k-1}})} \sum_{m \leq n} g(X_m) \\ &= \sum_{i \in I_{n,k}} f(X_i, ..., X_{i_{k-1}}) g(X_{i_k}) + \sum_{i \in I_{n,k-1}} \left[f(X_{i_1}, ..., X_{i_{k-1}}) \sum_{j=1}^{k-1} g(X_{i_j}) \right] \end{aligned} \tag{15}$$

Let $\phi_j(X_1, ..., X_{k-1}) = f(X_1, ..., X_{k-1}) g(X_j)$, $1 \leq j \leq k-1$ and $\phi(X_1, ..., X_k) = f(X_1, ..., X_{k-1}) g(X_k)$. Then,

$$n_{p_{k-1}} A_n(f) n A_n(g) = n_{p_k} A_n(\phi) + n_{p_{k-1}} \sum_{j=1}^{k-1} A_n(\phi_j)$$

Dividing by n_{p_k}, we have

$$\frac{n}{n-k+1} A_n(f) A_n(g) = A_n(\phi) + \frac{1}{n-k+1} \sum_{j=1}^{k} A_n(\phi_j)$$

by 15, and fact that $||f||_\infty, ||g||_\infty < \infty$, we have

$$\mathbb{E}[f(X_1, ..., X_{k-1}|\xi)]\mathbb{E}[g(X_1)|\xi] = \mathbb{E}[f(X_1, ..., X_{k-1}) g(X_k)|\xi] \tag{16}$$

Thus, we have using (16) that for any collection of bounded functions $f_1, ..., f_k$,

$$\mathbb{E}[\prod_{i=1}^{k} f_i(X_i)|\xi] = \prod_{i=1}^{k} \mathbb{E}[f_i(X_i)|\xi]$$

$\square$

Message: given the "symmetry" assumption and given "exchangeable" statistics, the underlying r.v. conditionally become i.i.d.!

A nice example. Let X_i be exchangeable r.v.s. taking values in $\{0, 1\}$. Then there exists distributions on $[0, 1]$ with distribution function F s.t.

$$\mathbb{P}(X_1 + ... + X_n = k) = \int_0^1 \theta^k (1-\theta)^{n-k} dF(\theta)$$

for all n. That is, mixture of i.i.d. r.v.

References

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