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Introduction to Ito calculus.

Content.

1. Spaces $\mathcal{L}_2, \mathcal{M}_2, \mathcal{M}_{2,c}$.
2. Quadratic variation property of continuous martingales.

1 Doob-Kolmogorov inequality. Continuous time version

Let us establish the following continuous time version of the Doob-Kolmogorov inequality. We use RCLL as abbreviation for right-continuous function with left limits.

Proposition 1. *Suppose $X_t \geq 0$ is a RCLL sub-martingale. Then for every $T, x \geq 0$*

$$\mathbb{P}(\sup_{0 \leq t \leq T} X_t \geq x) \leq \frac{\mathbb{E}[X_T^2]}{x^2}.$$

Proof. Consider any sequence of partitions $\Pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{N_n}^n = T\}$ such that $\Delta(\Pi_n) = \max_j |t_{j+1}^n - t_j^n| \rightarrow 0$. Additionally, suppose that the sequence Π_n is nested, in the sense the for every $n_1 \leq n_2$, every point in Π_{n_1} is also a point in Π_{n_2}. Let $X_t^n = X_{t_j^n}$ where $j = \max\{i : t_i \leq t\}$. Then X_t^n is a sub-martingale adopted to the same filtration (notice that this would not be the case if we instead chose right ends of the intervals). By the discrete version of the D-K inequality (see previous lectures), we have

$$\mathbb{P}(\max_{j \leq N_n} X_{t_j}^n \geq x) = \mathbb{P}(\sup_{t \leq T} X_t^n \geq x) \leq \frac{\mathbb{E}[X_T^2]}{x^2}.$$

By RCLL, we have $\sup_{t \leq T} X_t^n \rightarrow \sup_{t \leq T} X_t$ a.s. Indeed, fix $\epsilon > 0$ and find $t_0 = t_0(\omega)$ such that $X_{t_0} \geq \sup_{t \leq T} X_t - \epsilon$. Find n large enough and

$j = j(n)$ such that $t_{j(n)-1} \le t_0 \le t^n_{j(n)}$. Then $t_{j(n)} \to t_0$ as $n \to \infty$. By right-continuity of X, $X_{t_{j(n)}} \to X_{t_0}$. This implies that for sufficiently large n, $\sup_{t \le T} X^n_t \ge X_{t_{j(n)}} \ge X_{t_0} - 2\epsilon$, and the a.s. convergence is established. On the other hand, since the sequence Π_n is nested, then the sequence $\sup_{t \le T} X^n_t$ is non-decreasing. By continuity of probabilities, we obtain $\mathbb{P}(\sup_{t \le T} X^n_t \ge x) \to \mathbb{P}(\sup_{t \le T} X_t \ge x)$.

□

2 Stochastic processes and martingales

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t, t \in \mathbb{R}_+)$. We assume that all zero-measure events are "added" to $\mathcal{F}_0$. Namely, for every $A \subset \Omega$, such that for some $A' \in \mathcal{F}$ with $\mathbb{P}(A') = 0$ we have $A \subset A' \in \mathcal{F}$, then A also belongs to $\mathcal{F}_0$. A filtration is called right-continuous if $\mathcal{F}_t = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}$. From now on we consider exclusively right-continuous filtrations. A stochastic process X_t adopted to this filtration is a measurable function $X : \Omega \times [0, \infty) \to \mathbb{R}$, such that $X_t \in \mathcal{F}_t$ for every t. Denote by $\mathcal{L}_2$ the space of processes s.t. the Riemann integral $\int_0^T X_t(\omega) dt$ exists a.s. and moreover $\mathbb{E}[\int_0^T X_t^2 dt] < \infty$ for every $T > 0$. This implies $\mathbb{P}(\omega : \int_0^T |X_t(\omega)| dt < \infty, \forall T) = 1$.

Let $\mathcal{M}_2$ consist of square integrable right-continuous martingales with left limits (RCLL). Namely $\mathbb{E}[X_t^2] < \infty$ for every $X \in \mathcal{M}_2$ and $t \ge 0$. Finally $\mathcal{M}_{2,c} \subset \mathcal{M}_2$ is a further subset of processes consisting of a.s. continuous processes. For each $T > 0$ we define a norm on $\mathcal{M}_2$ by $\|X\| = \|X\|_T = (\mathbb{E}[X_T^2])^{1/2}$. Applying sub-martingale property of X_t^2 we have $\mathbb{E}[X_{T_1}^2] \le \mathbb{E}[X_{T_2}^2]$ for every $T_1 \le T_2$.

A stochastic process Y_t is called a *version* of X_t if for every $t \in \mathbb{R}_+, \mathbb{P}(X_t = Y_t) = 1$. Notice, this is weaker than saying $\mathbb{P}(X_t = Y_t, \forall t) = 1$.

Proposition 2. *Suppose $(X_t, \mathcal{F}_t)$ is a submartingale and $t \to \mathbb{E}[X_t]$ is a continuous function. Then there exists a version Y_t of X_t which is RCLL.*

We skip the proof of this fact.

Proposition 3. *$\mathcal{M}_2$ is a complete metric space and (w.r.t. $\|\cdot\|$) $\mathcal{M}_{2,c}$ is a closed subspace of $\mathcal{M}_2$.*

Proof. We need to show that if $X^{(n)} \in \mathcal{M}_2$ is Cauchy, then there exists $X \in \mathcal{M}_2$ with $\|X^{(n)} - X\| \to 0$.

Assume $X^{(n)}$ is Cauchy. Fix $t \le T$ Since $X^{(n)} - X^{(m)}$ is a martingale as well, $\mathbb{E}[(X_t^{(n)} - X_t^{(m)})^2] \le \mathbb{E}[(X_T^{(n)} - X_T^{(m)})^2]$. Thus $X_t^{(n)}$ is Cauchy as well. We know that the space $\mathbb{L}_2$ of random variables with finite second moment is closed. Thus for each t there exists a r.v. X_t s.t. $\mathbb{E}[(X_t^{(n)} - X_t)^2] \to 0$ as $n \to \infty$. We claim that since $X_t^{(n)} \in \mathcal{F}_t$ and $X^{(n)}$ is RCLL, then $(X_t, t \geq 0)$ is adopted to $\mathcal{F}_t$ as well (exercise). Let us show it is a martingale. First $\mathbb{E}[|X_t|] < \infty$ since in fact $\mathbb{E}[X_t^2] < \infty$. Fix $s < t$ and $A \in \mathcal{F}_s$. Since each $X_t^{(n)}$ is a martingale, then $\mathbb{E}[X_t^{(n)} 1(A)] = \mathbb{E}[X_s^{(n)} 1(A)]$. We have

$$\mathbb{E}[X_t 1(A)] - \mathbb{E}[X_s 1(A)] = \mathbb{E}[(X_t - X_t^{(n)})1(A)] - \mathbb{E}[(X_s - X_s^{(n)})1(A)]$$

We have $\mathbb{E}[|X_t - X_t^{(n)}|1(A)] \leq \mathbb{E}[|X_t - X_t^{(n)}|] \leq (\mathbb{E}[(X_t - X_t^{(n)})^2])^{1/2} \to 0$ as $n \to \infty$. A similar statement holds for s. Since the left-hand side does not depend on n, we conclude $\mathbb{E}[X_t 1(A)] = \mathbb{E}[X_s 1(A)]$ implying $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$, namely X_t is a martingale. Since $\mathbb{E}[X_t] = \mathbb{E}[X_0]$ is constant and therefore continuous as a function of t, then there exists version of X_t which is RCLL. For simplicity we denote it by X_t as well. We constructed a process $X_t \in \mathcal{M}_2$ s.t. $\mathbb{E}[(X_t^{(n)} - X_t)^2] \to 0$ for all $t \leq T$. This proves completeness of $\mathcal{M}_2$.

Now we deal with closeness of $\mathcal{M}_{2,c}$. Since $X_t^{(n)} - X_t$ is a martingale, $(X_t^{(n)} - X_t)^2$ is a submartingale. Since $X_t \in \mathcal{M}_2$, then $(X_t^{(n)} - X_t)^2$ is RCLL. Then submartingale inequality applies. Fix $\epsilon > 0$. By submartingale inequality we have

$$\mathbb{P}(\sup_{t \leq T} |X_t^{(n)} - X_t| > \epsilon) \leq \frac{1}{\epsilon^2} \mathbb{E}[(X_T^{(n)} - X_T)^2] \to 0,$$

as $n \to \infty$. Then we can choose subsequence n_k such that

$$\mathbb{P}(\sup_{t \leq T} |X_t^{(n_k)} - X_t| > 1/k) \leq \frac{1}{2^k}.$$

Since $1/2^k$ is summable, by Borel-Cantelli Lemma we have $\sup_{t \leq T} |X_t^{(n_k)} - X_t| \to 0$ almost surely: $\mathbb{P}(\{\omega \in \Omega : \sup_{t \leq T} |X_t^{(n_k)}(\omega) - X_t(\omega)| \to 0\}) = 1$. Recall that a uniform limit of continuous functions is continuous as well (first lecture). Thus X_t is continuous a.s. As a result $X_t \in \mathcal{M}_{2,c}$ and $\mathcal{M}_{2,c}$ is closed. $\square$

3 Doob-Meyer decomposition and quadratic variation of processes in $\mathcal{M}_{2,c}$

Consider a Brownian motion B_t adopted to a filtration $\mathcal{F}_t$. Suppose this filtration makes B_t a strong Markov process (for example $\mathcal{F}_t$ is generated by B itself). Recall that both B_t and $B_t^2 - t$ are martingales and also $B \in \mathcal{M}_{2,c}$. Finally recall that the quadratic variation of B over any interval $[0, t]$ is t. There is a generalization of these observations to processes in $\mathcal{M}_{2,c}$. For this we need to recall the following result.

Theorem 1 (**Doob-Meyer decomposition**). *Suppose* $(X_t, \mathcal{F}_t)$ *is a continuous non-negative sub-martingale. Then there exist a continuous martingale* M_t *and a.s. non-decreasing continuous process* A_t *with* $A_0 = 0$*, both adopted go* $\mathcal{F}_t$ *such that* $X_t = A_t + M_t$*. The decomposition is unique in the almost sure sense.*

The proof of this theorem is skipped. It is obtained by appropriate discretization and passing to limits. The discrete version of this result we did earlier. See [1] for details.

Now suppose $X_t \in \mathcal{M}_{2,c}$. Then X_t^2 is a continuous non-negative submartingale and thus DM theorem applies. The part A_t in the unique decomposition of X_t^2 is called *quadratic variation* of X_t (we will shortly justify this) and denoted $\langle X_t \rangle$.

Theorem 2. *Suppose* $X_t \in \mathcal{M}_{2,c}$*. Then for every* $t > 0$ *the following convergence in probability takes place*

$$\lim_{\Pi_n : \Delta(\Pi_n) \to 0} \sum_{0 \le j \le n-1} (X_{t_{j+1}} - X_{t_j})^2 \to \langle X_t \rangle,$$

where the limit is over all partitions $\Pi_n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ *and* $\Delta(\Pi_n) = \max_j |t_j - t_{j-1}|$.

Proof. Fix $s < t$. Let $X \in \mathcal{M}_{2,c}$. We have

$$\begin{aligned}
\mathbb{E}[(X_t - X_s)^2 - (\langle X_t \rangle - \langle X_s \rangle)|\mathcal{F}_s] &= \mathbb{E}[X_t^2 - 2X_tX_s + X_s^2 - (\langle X_t \rangle - \langle X_s \rangle)|\mathcal{F}_s] \\
&= \mathbb{E}[X_t^2|\mathcal{F}_s] - 2X_s\mathbb{E}[X_t|\mathcal{F}_s] + X_s^2 - \mathbb{E}[\langle X_t \rangle|\mathcal{F}_s] + \langle X_s \rangle \\
&= \mathbb{E}[X_t^2 - \langle X_t \rangle|\mathcal{F}_s] - X_s^2 + \langle X_s \rangle \\
&= 0.
\end{aligned}$$

Thus for every $s < t \le u < v$ by conditioning first on $\mathcal{F}_u$ and using tower property we obtain

$$\mathbb{E}\Big((X_t - X_s)^2 - (\langle X_t \rangle - \langle X_s \rangle)\Big)\Big((X_u - X_v)^2 - (\langle X_u \rangle - \langle X_v \rangle)\Big) = 0 \quad (1)$$

The proof of the following lemma is application of various "carefully placed" tower properties and is omitted. See [1] Lemma 1.5.9 for details.

Lemma 1. *Suppose* $X \in \mathcal{M}_2$ *satisfies* $|X_s| \le M$ *a.s. for all* $s \le t$*. Then for every partition* $0 = t_0 \le \cdots \le t_n = t$

$$\mathbb{E}\Big(\sum_j (X_{t_{j+1}} - X_{t_j})^2\Big)^2 \le 6M^4.$$

Lemma 2. *Suppose $X \in \mathcal{M}_2$ satisfies $|X_s| \le M$ a.s. for all $s \le t$. Then*

$$\lim_{\Delta(\Pi_n)\to 0} \mathbb{E}[\sum_j (X_{t_{j+1}} - X_{t_j})^4] = 0,$$

where $\Pi_n = \{0 = t_0 < \cdots < t_n = t\}, \Delta(\Pi_n) = \max_j |t_{j+1} - t_j|$.

Proof. We have

$$\sum_j (X_{t_{j+1}} - X_{t_j})^4 \le \sum_j (X_{t_{j+1}} - X_{t_j})^2 \sup\{|X_r - X_s|^2 : |r - s| \le \Delta(\Pi_n)\}.$$

Applying Cauchy-Schwartz inequality and Lemma 1 we obtain

$$\begin{aligned}\Big(\mathbb{E}[\sum_j (X_{t_{j+1}} - X_{t_j})^4]\Big)^2 &\le \mathbb{E}\Big(\sum_j (X_{t_{j+1}} - X_{t_j})^2\Big)^2 \mathbb{E}[\sup\{|X_r - X_s|^4 : |r - s| \le \Delta(\Pi_n)\}] \\ &\le 6M^4 \mathbb{E}[\sup\{|X_r - X_s|^4 : |r - s| \le \Delta(\Pi_n)\}].\end{aligned}$$

Now $X(\omega)$ is a.s. continuous and therefore uniformly continuous on $[0, t]$. Therefore, a.s. $\sup\{|X_r - X_s|^2 : |r - s| \le \Delta(\Pi_n)\} \to 0$ as $\Delta(\Pi_n) \to 0$. Also $|X_r - X_s| \le 2M$ a.s. Applying Bounded Convergence Theorem, we obtain that $\mathbb{E}[\sup\{|X_r - X_s|^4 : |r - s| \le \Delta(\Pi_n)\}]$ converges to zero as well and the result is obtained. □

We now return to the proof of the proposition. We first assume $|X_s| \le M$ and $\langle X_s \rangle \le M$ a.s. for $s \in [0, t]$.

We have (using a telescoping sum)

$$\mathbb{E}\Big(\sum_j (X_{t_{j+1}} - X_{t_j})^2 - \langle X_t \rangle\Big)^2 = \mathbb{E}\Big(\sum_j \big((X_{t_{j+1}} - X_{t_j})^2 - (\langle X_{t_{j+1}} \rangle - \langle X_{t_j} \rangle)\big)\Big)^2$$

When we expand the square the terms corresponding to cross products with $j_1 \ne j_2$ disappear due to (1). Thus the expression is equal to

$$\begin{aligned}&\mathbb{E}\sum_j \Big((X_{t_{j+1}} - X_{t_j})^2 - (\langle X_{t_{j+1}} \rangle - \langle X_{t_j} \rangle)\Big)^2 \\ &\qquad \le 2\mathbb{E}\Big[\sum_j (X_{t_{j+1}} - X_{t_j})^4\Big] + 2\mathbb{E}[\sum_j (\langle X_{t_{j+1}} \rangle - \langle X_{t_j} \rangle)^2].\end{aligned}$$

The first term converges to zero as $\Delta(\Pi_n) \to 0$ by Lemma 2.

We now analyze the second term. Since $\langle X_t \rangle$ is a.s. non-decreasing, then

$$\sum_j (\langle X_{t_{j+1}} \rangle - \langle X_{t_j} \rangle)^2 \le \sum_j (\langle X_{t_{j+1}} \rangle - \langle X_{t_j} \rangle) \sup_{0 \le s \le r \le t} \{ \langle X_r \rangle - \langle X_s \rangle : |r - s| \le \Delta(\Pi_n) \}$$

Thus the expectation is upper bounded by

$$\mathbb{E}[\langle X_t \rangle \sup_{0 \le s \le r \le t} \{ \langle X_r \rangle - \langle X_s \rangle : |r - s| \le \Delta(\Pi_n) \}] \tag{2}$$

Now $\langle X_t \rangle$ is a.s. continuous and thus the supremum term converges to zero a.s. as $n \to \infty$. On the other hand a.s. $\langle X_t \rangle (\langle X_r \rangle - \langle X_s \rangle) \le 2M^2$. Thus using Bounded Convergence Theorem, we obtain that the expectation in (2) converges to zero as well. We conclude that in the bounded case $|X_s|, \langle X_s \rangle \le M$ on $[0, t]$, the quadratic variation of X_s over $[0, t]$ converges to $\langle X_t \rangle$ in $\mathbb{L}_2$ sense. This implies convergence in probability as well.

It remains to analyze the general (unbounded) case. Introduce stopping times T_M for every $M \in \mathbb{R}_+$ as follows

$$T_M = \min\{ t : |X_t| \ge M \ \text{ or } \ \langle X_t \rangle \ge M \}$$

Consider $X_t^M \triangleq X_{t \wedge T_M}$. Then $X^M \in \mathcal{M}_{2,c}$ and is a.s. bounded. Further since $X_t^2 - \langle X_t \rangle$ is a martingale, then $X_{t \wedge T_M}^2 - \langle X_{t \wedge T_M} \rangle$ is a bounded martingale. Since Doob-Meyer decomposition is unique, we that $\langle X_{t \wedge T_M} \rangle$ is indeed the unique non-decreasing component of the stopped martingale $X_{t \wedge T_M}$. There is a subtlety here: X_t^M is a continuous martingale and therefore it has its own quadratic variation $\langle X_t^M \rangle$ - the unique non-decreasing a.s. process such that $(X_t^M)^2 - \langle X_t^M \rangle$ is a martingale. It is a priori non obvious that $\langle X_t^M \rangle$ is the same as $\langle X_{t \wedge T_M} \rangle$ - quadratic variation of X_t stopped at T_M. But due to uniqueness of the D-M decomposition, it is.

Fix $\epsilon > 0, t \ge 0$ and find M large enough so that $\mathbb{P}(T_M < t) < \epsilon/2$. This is possible since X_t and $\langle X_t \rangle$ are continuous processes. Now we have

$$\begin{aligned}
&\mathbb{P}\Big(\Big| \sum_j (X_{t_{j+1}} - X_{t_j})^2 - \langle X_t \rangle \Big| > \epsilon \Big) \\
&\quad \le \mathbb{P}\Big(\Big| \sum_j (X_{t_{j+1}} - X_{t_j})^2 - \langle X_t \rangle \Big| > \epsilon, t \le T_M \Big) + \mathbb{P}(T_M < t) \\
&\quad = \mathbb{P}\Big(\Big| \sum_j (X_{t_{j+1} \wedge T_M} - X_{t_j \wedge T_M})^2 - \langle X_{t \wedge T_M} \rangle \Big| > \epsilon, t \le T_M \Big) + \mathbb{P}(T_M < t) \\
&\quad \le \mathbb{P}\Big(\Big| \sum_j (X_{t_{j+1} \wedge T_M} - X_{t_j \wedge T_M})^2 - \langle X_{t \wedge T_M} \rangle \Big| > \epsilon \Big) + \mathbb{P}(T_M < t).
\end{aligned}$$

We already established the result for bounded martingales and quadratic variations. Thus, there exists $\delta = \delta(\epsilon) > 0$ such that, provided $\Delta(\Pi) < \delta$, we have

$$\mathbb{P}\Big(\Big|\sum_j (X_{t_{j+1}\wedge T_M} - X_{t_j \wedge T_M})^2 - \langle X_{t\wedge T_M}\rangle\Big| > \epsilon\Big) < \epsilon/2.$$

We conclude that for $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ with $\Delta(\Pi) < \delta$, we have

$$\mathbb{P}\Big(\Big|\sum_j (X_{t_{j+1}} - X_{t_j})^2 - \langle X_t\rangle\Big| > \epsilon\Big) < \epsilon.$$

□

4 Additional reading materials

- Chapter I. Karatzas and Shreve [1]

References

[1] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, Springer, 1991.

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