MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.265/15.070J Fall 2013
Lecture 15 10/30/2013

Ito integral for simple processes

Content.

1. Simple processes. Ito isometry
2. First 3 steps in constructing Ito integral for general processes

1 Ito integral for simple processes. Ito isometry

Consider a Brownian motion B_t adopted to some filtration $\mathcal{F}_t$ such that $(B_t, \mathcal{F}_t)$ is a strong Markov process. As an example we can take filtration generated by the Brownian motion itself. Our goal is to give meaning to expressions of the form $\int X_t dB_t = \int X_t(\omega) dB_t(\omega)$, where X_t is some stochastic process which is adapted to the same filtration as B_t. We will primarily deal with the case $X \in \mathcal{L}_2$, although it is possible to extend definitions to more general processes using the notion of *local martingales*. As in the case of usual integration, the idea is to define $\int X_t(\omega) dB_t(\omega)$ as some kind of a limit of (random) sums $\sum_j X_{t_j}(\omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega))$ and show that the limit exists in some appropriate sense. As X_t we can take all kinds of processes, including B_t itself. For example we will show that $\int_0^T B_t dB_t$ makes sense and equals $(1/2)B_T^2 - (1/2)T$.

Definition 1. *A process $X \in \mathcal{L}_2 = \mathcal{L}_2(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is called* simple *if there exists a countable partition $\Pi : 0 = t_0 < \cdots < t_n < \cdots$ with $\lim_n t_n = \infty$ such that $X_t(\omega) = X_{t_j}(\omega)$ for all $t \in [t_j, t_{j+1}), j = 0, 1, 2, \ldots$ for all $\omega \in \Omega$. The subspace of simple processes is denoted by $\mathcal{L}_2^0$*

We assume that partition is such that $t_j \to \infty$ as $j \to \infty$. It is important to note that we assume that the partition Π does not depend on ω. Thus not every piece-wise constant process is a simple process. Give an example of a piece-wise constant process which is not simple. Note that since $X_t \in \mathcal{F}_t$ we have $X_{t_j} \in \mathcal{F}_{t_j}$ for each j. As an example of simple process, fix any partition Π and a process $X_t \in \mathcal{L}_2$ and consider the process $\hat{X}_t(\omega)$ defined by $\hat{X}_t(\omega) =$

$X_{t_j}(\omega)$, where t_j is defined by $t \in [t_j, t_{j+1})$. In the definition it is important that $\hat{X}_t = X_{t_j}$ and not $X_{t_{j+1}}$. Observe that the latter is not necessarily adopted to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

Given a simple process X and t, define its integral by

$$I_t(X(\omega)) = \sum_{0\le j\le n-1} X_{t_j}(\omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)) + X_{t_n}(\omega)(B_t(\omega) - B_{t_n}(\omega)),$$

where $n = \max\{j : t_j \le t\}$. Observe that $I_t(X)$ is an a.s. continuous function (as B_t is a.s. continuous).

Theorem 1. *The following properties hold for $I_t(X)$*

$$I_t(\alpha X + \beta Y) = \alpha I_t(X) + \beta I_t(Y). \tag{1}$$

$$\mathbb{E}[I_t^2(X)] = \mathbb{E}[\int_0^t X_s^2 ds] \ \textbf{[Ito isometry]}, \tag{2}$$

$$I_t(X) \in \mathcal{M}_{2,c}, \tag{3}$$

$$\mathbb{E}[(I_t(X) - I_s(X))^2|\mathcal{F}_s] = \mathbb{E}[\int_s^t X_u^2 du],\ \forall\, 0 \le s < t \le T. \tag{4}$$

Notice that (4) is a generalization of Ito isometry. We only prove Ito isometry, the proof of (4) follows along the same lines.

Proof. Define $t_n = t$ for convenience. We begin with (1). Let $\{t_j^1\}$ and $\{t_j^2\}$ be partitions corresponding to simple processes X and Y. Consider a partition $\{t_j\}$ obtained as a union of these two partitions. For each t_j which belongs to the second partition but not the first define $X_{t_j} = X_{t_i^1}$, where t_i^1 is the largest point not exceeding t_j. Do a similar thing for Y. Observe that now $X_t = X_{t_j}$ for $t \in [t_j, t_{j+1})$. The linearity of Ito integral then follows straight from the definition.

Now for (2) we have

$$\mathbb{E}[I_t^2(X)] = \sum_{0\le j_1,j_2\le n-1} \mathbb{E}[X_{t_{j_1}} X_{t_{j_2}} (B_{t_{j_1+1}} - B_{t_{j_1}})(B_{t_{j_2+1}} - B_{t_{j_2}})].$$

When $j_1 < j_2$ we have

$$\mathbb{E}[X_{t_{j_1}} X_{t_{j_2}} (B_{t_{j_1+1}} - B_{t_{j_1}})(B_{t_{j_2+1}} - B_{t_{j_2}})] = 0$$

which we obtain by conditioning on $\mathcal{F}_{t_{j_2}}$, using the tower property and observing that all of the random variables involved except for $B_{t_{j_2+1}}$ are measurable with respect to $\mathcal{F}_{t_{j_2}}$ (recall that $\mathcal{F}_{t_{j_1}} \subset \mathcal{F}_{t_{j_2}}$).

Now when $j_1 = j_2 = j$ we have

$$\begin{aligned}\mathbb{E}[X_{t_j}^2(B_{t_{j+1}} - B_{t_j})^2] &= \mathbb{E}[X_{t_j}^2 \mathbb{E}[(B_{t_{j+1}} - B_{t_j})^2 | \mathcal{F}_{t_j}]] \\ &= \mathbb{E}[X_{t_j}^2 (t_{j+1} - t_j)].\end{aligned}$$

Combining, we obtain

$$\mathbb{E}[I_t^2(X)] = \sum_j \mathbb{E}[X_{t_j}^2(t_{j+1} - t_j)] = \mathbb{E}[\sum_j X_{t_j}^2(t_{j+1} - t_j)] = \mathbb{E}[\int_0^t X_s^2 ds].$$

Let us show (3). We already know that the process $I_t(X)$ is continuous. From Ito isometry it follows that $\mathbb{E}[I_t^2(X)] < \infty$. It remains to show that it is a martingale. Thus fix $s < t$. Define $t_n = t$ and define $j_0 = \max\{j : t_j \leq s\}$.

$$\begin{aligned}\mathbb{E}[I_t(X)|\mathcal{F}_s] &= \mathbb{E}[\sum_{j \leq n-1} X_{t_j}(B_{t_{j+1}} - B_{t_j})|\mathcal{F}_s] \\ &= \mathbb{E}[\sum_{j \leq j_0 - 1} X_{t_j}(B_{t_{j+1}} - B_{t_j})|\mathcal{F}_s] + \mathbb{E}[X_{t_{j_0}}(B_s - B_{t_{j_0}})|\mathcal{F}_s] \\ &+ \mathbb{E}[X_{t_{j_0}}(B_{t_{j_0+1}} - B_s)|\mathcal{F}_s] + \mathbb{E}[\sum_{j > j_0} X_{t_j}(B_{t_{j+1}} - B_{t_j})|\mathcal{F}_s] \\ &= \mathbb{E}[\sum_{j \leq j_0 - 1} X_{t_j}(B_{t_{j+1}} - B_{t_j})|\mathcal{F}_s] + \mathbb{E}[X_{t_{j_0}}(B_s - B_{t_{j_0}})|\mathcal{F}_s] \\ &= I_s(X).\end{aligned}$$

(think about justifying last two equalities).

□

2 Constructing Ito integral for general square integrable processes

The idea for defining Ito integral $\int X dB$ for general processes in $\mathcal{L}_2$ is to approximate X by simple processes $X^{(n)}$ and define $\int X dB$ as a limit of $\int X^{(n)} dB$, which we have already defined.

For this purpose we need to show that we can indeed approximate X with simple processes appropriately. We do this in 3 steps.

Step 1.

Proposition 1. *Suppose $X \in \mathcal{L}_2$ is an a.s. bounded continuous process in the sense $\exists M$ s.t. $\mathbb{P}(\omega : \sup_{t \geq 0} |X_t(\omega)| \leq M) = 1$. Then for every $T > 0$ there* *exists a sequence of simple processes $X^n \in \mathcal{L}_2^0$ such that*

$$\lim_n \mathbb{E}[\int_0^T (X_t^n - X_t)^2 dt] = 0. \tag{5}$$

Proof. Fix a sequence of partitions $\Pi_n = \{t_j^n\}$ of $[0, T]$ such that $\Delta_n = \max(t_{j+1}^n - t_j^n) \to 0$ as $n \to \infty$. Given process X, consider the modified process $X_t^n = X_{t_j^n}$ for all $t \in [t_j^n, t_{j+1}^n)$. This process is simple and is adapted to $\mathcal{F}_t$. Since X is a.s. continuous, then a.s. $X_t(\omega) = \lim_{n\to\infty} X_t^n(\omega)$ (notice that we are using left-continuity part of continuity). We conclude that a sequence of measurable functions $X^n : \Omega \times [0, T] \to \mathbb{R}$ a.s. converges to $X : \Omega \times [0, T] \to \mathbb{R}$. On the other hand $\mathbb{P}(\omega : \sup_{t\le T} |X_t^n(\omega)| \le M) = 1$. Using Bounded Convergence Theorem, the a.s. convergence extends to integrals: $\mathbb{E}[\int_0^T (X_t^n - X_t)^2 dt] \to 0$. □

Step 2.

Proposition 2. *Suppose $X \in \mathcal{L}_2$ is a bounded, but not necessarily continuous process: $|X| \le M$ a.s. For every $T > 0$, there exists a sequence of a.s. bounded continuous processes X_n such that*

$$\lim_n \mathbb{E}[\int_0^T (X_t^n - X_t)^2 dt] = 0. \tag{6}$$

Proof. We use a certain "regularization" trick to turn a bounded process into a bounded continuous approximation. Let $X_t^n = n \int_{t-1/n}^t X_s ds$. We have $|X^n| \le n(1/n)M = M$ and $|X_{t'}^n - X_t^n| \le 2n|t' - t|M$ (verify this), implying that X_t^n is a.s. bounded continuous. Since X_t is a.s. Riemann integrable, then for almost all ω, the set of discontinuity points of of $X_t(\omega)$ has measure zero and for all continuity points t by Fundamental Theorem of Calculus, we have $\lim_{n\to\infty} X_t^n(\omega) = X_t(\omega)$. We conclude that $X^n : \Omega \times [0, T] \to \mathbb{R}$ converges a.s. to X on the same domain. Applying the Bounded Convergence Theorem we obtain the result. □

Step 3.

Proposition 3. *Suppose $X \in \mathcal{L}_2$. For every $T > 0$ there exists a sequence of a.s. bounded processes $X_n \in \mathcal{L}_2$ such that*

$$\lim_n \mathbb{E}[\int_0^T (X_t^n - X_t)^2 dt] = 0. \tag{7}$$

Proof. Define X^n by $X^n_t = X_t$ when $-n \leq X_t \leq n$, $X^n_t = -n$, when $X_t < -n$ and $X^n_t = n$, when $X_t > n$. We have $X^n \to X$ a.s. w.r.t both ω and $t \in [0, T]$. Also $|X^n_t| \leq |X_t|$ implying

$$\begin{aligned}\int_0^T (X^n_t - X_t)^2 dt &\leq 2\int_0^T (X^n_t)^2 dt + 2\int_0^T X_t^2 dt \\ &\leq 4\int_0^T X_t^2 dt.\end{aligned}$$

Since $\mathbb{E}[\int_0^T X_t^2 dt] < \infty$, then applying Dominated Convergence Theorem, we obtain the result.

Exercise 1. *Establish (7) by applying instead Monotone Convergence Theorem.*

□

3 Additional reading materials

- Karatzas and Shreve [1].
- Øksendal [2], Chapter III.

References

[1] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, Springer, 1991.

[2] B. Øksendal, *Stochastic differential equations*, Springer, 1991.

15.070J / 6.265J Advanced Stochastic Processes
Fall 2013