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Ito integral. Properties

Content.

1. Definition of Ito integral
2. Properties of Ito integral

1 Ito integral. Existence

We continue with the construction of Ito integral. Combining the results of Propositions 1-3 from the previous lecture we proved the following result.

Proposition 1. *Given any process $X \in \mathcal{L}_2$ there exists a sequence of simple processes $X_n \in \mathcal{L}_2^0$ such that*

$$\lim_{n\to\infty} \mathbb{E}[\int_0^T (X_n(t) - X(t))^2 dt] = 0. \tag{1}$$

Now, given a process $X \in \mathcal{L}_2$, we fix any sequence of simple processes $X^n \in \mathcal{L}_2^0$ which satisfies (1) for a given T. Recall, that we already have defined Ito integral for simple processes $I_t(X^n)$.

Proposition 2. *Suppose a sequence of simple processes X^n satisfies (1). There exists a process $Z_t \in M_{2,c}$ satisfying $\lim_n \mathbb{E}[(Z_t - I_t(X^n))^2] = 0$ for all $0 \le t \le T$. This process is unique a.s. in the following sense: if $\hat{X}_t^n$ is another process satisfying (1) and $\hat{Z}$ is the corresponding limit, then $\mathbb{P}(\hat{Z}_t = Z_t, \forall t \in [0,T]) = 1$.*

Proof. Fix $T > 0$. Applying linearity of $I_t(X)$ and Ito isometry

$$\begin{aligned}
\mathbb{E}[(I_T(X_m) - I_T(X_n))^2] &= \mathbb{E}[I_T^2(X_m - X_n)] \\
&= \mathbb{E}[\int_0^T (X_m(t) - X_n(t))^2 dt] \\
&\le 2\mathbb{E}[\int_0^T (X(t) - X_m(t))^2 dt] + 2\mathbb{E}[\int_0^T (X(t) - X_n(t))^2 dt].
\end{aligned}$$

But since the sequence X_n satisfies (1), it follows that the sequence $I_T(X^n)$ is Cauchy in $\mathbb{L}_2$ sense. Recall now from Theorem 2.2. previous lecture that each $I_t(X^n)$ is a continuous square integrable martingale: $I_t(X^n) \in M_{2,c}$ Applying Proposition 2, Lecture 1, which states that $M_{2,c}$ is a closed space, there exists a limit $Z_t, t \in [0,T]$ in $M_{2,c}$ satisfying $\mathbb{E}[(Z_T - I_T(X^n))^2] \to 0$. The same applies to every $t \leq T$ since $(Z_t - I_t(X^n))^2$ is a submartingale.

It remains to show that such a process Z_t is unique. If $\hat{Z}_t$ is a limit of some sequence $\hat{X}^n$ satisfying (1), then by submartingale inequality for every $\epsilon > 0$ we have $\mathbb{P}(\sup_{t \leq T} |Z_t - \hat{Z}_t| \geq \epsilon) \leq \mathbb{E}[(Z_T - \hat{Z}_T)^2]/\epsilon^2$. But

$$\begin{aligned}\mathbb{E}[(Z_T - \hat{Z}_T)^2] \leq 3\mathbb{E}[(Z_T - I_T(X^n))^2] + 3\mathbb{E}[(I_T(X^n) - I_T(\hat{X}^n))^2] \\ + 3\mathbb{E}[(I_T(\hat{X}^n) - \hat{Z}_T)^2],\end{aligned}$$

and the right-hand side converges to zero. Thus $\mathbb{E}[(Z_T - \hat{Z}_T)^2] = 0$. It follows that $Z_t = \hat{Z}_t$ a.s. on $[0,T]$. Since T was arbitrary we obtain an a.s. unique limit on $\mathbb{R}_+$. □

Now we can formally state the definition of Ito integral.

Definition 1 (Ito integral). *Given a stochastic process $X_t \in \mathcal{L}_2$ and $T > 0$, its Ito integral $I_t(X), t \in [0,T]$ is defined to be the unique process Z_t constructed in Proposition 2.*

We have defined Ito integral as a process which is defined only on a finite interval $[0,T]$. With a little bit of extra work it can be extended to a process $I_t(X)$ defined for all $t \geq 0$, by taking $T \to \infty$ and taking appropriate limits. Details can be found in [1] and are omitted, as we will deal exclusively with Ito integrals defined on a finite interval.

2 Ito integral. Properties

2.1 Simple example

Let us compute the Ito integral for a special case $X_t = B_t$. We will do this directly from the definition. Later on we will develop calculus rules for computing the Ito integral for many interesting cases.

We fix a sequence of partitions $\Pi_n : 0 = t_0 < \cdots < t_n = T$ and consider $B_t^n = B_{t_j}, t \in [t_j, t_{j+1})$. Assume that $\lim_n \Delta(\Pi_n) = 0$, where $\Delta(\Pi_n) = \max_j |t_{j+1} - t_j|$. We first show that this is sufficient for having

$$\lim_n \mathbb{E}[\int_0^T (B_t - B_t^n)^2 dt] = 0. \quad (2)$$

Indeed

$$\int_0^T (B_t - B_t^n)^2 dt = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (B_t - B_{t_j})^2 dt.$$

We have

$$\begin{aligned}\mathbb{E}[\int_{t_j}^{t_{j+1}} (B_t - B_{t_j})^2 dt &= \int_{t_j}^{t_{j+1}} \mathbb{E}[(B_t - B_{t_j})^2] dt \\ &= \int_{t_j}^{t_{j+1}} (t - t_j) dt \\ &= \frac{(t_{j+1} - t_j)^2}{2},\end{aligned}$$

implying

$$\mathbb{E}[\int_0^T (B_t - B_t^n)^2 dt] = \frac{1}{2}\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2 \le \Delta(\Pi_n)\sum_{j=0}^{n-1}(t_{j+1} - t_j) = \Delta(\Pi_n)T \to 0,$$

as $n \to \infty$. Thus (2) holds.

Thus we need to compute the $\mathbb{L}_2$ limit of

$$I_T(B_n) = \sum_j B_{t_j}(B_{t_{j+1}} - B_{t_j})$$

as $n \to \infty$. We use the identity

$$B_{t_{j+1}}^2 - B_{t_j}^2 = (B_{t_{j+1}} - B_{t_j})^2 + 2B_{t_j}(B_{t_{j+1}} - B_{t_j}),$$

implying

$$B^2(T) - B^2(0) = \sum_{j=0}^{n-1} B_{t_{j+1}}^2 - B_{t_j}^2 = \sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 + 2\sum_{j=0}^{n-1} B_{t_j}(B_{t_{j+1}} - B_{t_j}),$$

But recall the quadratic variation property of the Brownian motion:

$$\lim_n \sum_{j=0}^{n-1}(B_{t_{j+1}} - B_{t_j})^2 = T$$

in $\mathbb{L}_2$ (recall that the only requirement for this convergence was that $\Delta(\Pi_n) \to 0$). Therefore, also in $\mathbb{L}_2$

$$\sum_{j=0}^{n-1} B_{t_j}(B_{t_{j+1}} - B_{t_j}) \to \frac{1}{2}B^2(T) - \frac{T}{2}.$$

We conclude

Proposition 3. *The following identity holds*

$$I_T(B) = \int_0^T B_t dB_t = \frac{1}{2}B^2(T) - \frac{T}{2}.$$

Further, recall that since $B_t \in M_{2,c}$ then it admits a unique Doob-Meyer decomposition $B_t^2 = t + M_t$, where $t = \langle B_t \rangle$ is the quadratic variation of B_t and M_t is a continuous martingale. Thus we recognize M_t to be $2I_t(B)$.

2.2 Properties

We already know that $I_t(X) \in \mathcal{M}_{2,c}$, in particular it a is continuous martingale. Let us establish additional properties, some of which are generalizations of Theorem 2.2 from the previous lecture.

Proposition 4. *The following properties hold for $I_t(X)$:*

$$I_t(\alpha X + \beta Y) = \alpha I_t(X) + \beta I_t(Y), \ \ \forall \alpha, \beta, \tag{3}$$

$$\mathbb{E}[(I_t(X) - I_s(X))^2 | \mathcal{F}_s] = \mathbb{E}[\int_s^t X_u^2 du | \mathcal{F}_s], \forall\, 0 \le s < t \le T. \tag{4}$$

Furthermore, the quadratic variation of $I_t(X)$ on $[0,T]$ is $\int_0^T X_t^2 dt$.

Proof. The proof of (3) is straightforward and is skipped. We now prove (4). Fix any set $A \in \mathcal{F}_s$. We need to show that

$$\mathbb{E}[(I_t(X) - I_s(X))^2 I(A)] = \mathbb{E}[I(A) \int_s^t X_u^2 du].$$

Fix a sequence $X^n \in \mathcal{L}_2^0$ satisfying (1). Then

$$\begin{aligned}\mathbb{E}[(I_t(X) - I_s(X))^2 I(A)] = &\ \mathbb{E}[(I_t(X) - I_t(X^n))^2 I(A)] + \mathbb{E}[(I_t(X^n) - I_s(X^n))^2 I(A)] \\ &+ \mathbb{E}[(I_s(X^n) - I_s(X))^2 I(A)] \\ &+ 2\mathbb{E}(I_t(X) - I_t(X^n))(I_t(X^n) - I_s(X^n))I(A) \\ &+ 2\mathbb{E}(I_t(X) - I_t(X^n))(I_s(X^n) - I_s(X))I(A) \\ &+ 2\mathbb{E}(I_t(X^n) - I_s(X^n))(I_s(X^n) - I_s(X))I(A)\end{aligned}$$

But $\mathbb{E}[(I_t(X) - I_t(X^n))^2 I(A)] \to 0$ since $\mathbb{E}[(I_t(X) - I_t(X^n))^2] \to 0$ (definition of Ito integral). Similarly $\mathbb{E}[(I_s(X) - I_s(X^n))^2 I(A)] \to 0$. Applying Cauchy-Schwartz inequality

$$\begin{aligned}&|\mathbb{E}(I_t(X) - I_t(X^n))(I_t(X^n) - I_s(X^n))I(A)| \\ &\qquad \le (\mathbb{E}[(I_t(X) - I_t(X^n))^2])^{1/2}(\mathbb{E}[(I_t(X^n) - I_s(X^n))^2])^{1/2} \to 0\end{aligned}$$

from the definition of $I_t(X)$. Similarly we show that all the other terms with factor 2 in front converge to zero.

By property (2.6) Theorem 2.2 previous lecture, we have

$$\mathbb{E}[(I_t(X^n) - I_s(X^n))^2 I(A)] = \mathbb{E}[I(A)\int_s^t (X_u^n)^2 du]$$

Now

$$\begin{aligned}
\mathbb{E}[I(A)\int_s^t (X_u^n)^2 du] - \mathbb{E}[I(A)\int_s^t X_u^2 du] &= \mathbb{E}[I(A)\int_s^t (X_u^n - X_u)(X_u^n + X_u) du] \\
&\le \mathbb{E}[\int_s^t |(X_u^n - X_u)(X_u^n + X_u)| du] \\
&\le \mathbb{E}^{\frac{1}{2}}[\int_s^t (X_u^n - X_u)^2 du]\mathbb{E}^{\frac{1}{2}}[\int_s^t (X_u^n + X_u)^2 du]
\end{aligned}$$

where Cauchy-Schwartz inequality was used in the last step. Now the first term in the product converges to zero by the assumption (1) and the second is uniformly bounded in n (exercise). The assertion then follows.

Now we prove the last part. Applying Proposition 3 from Lecture 1, it suffices to show that $I_t^2(X) - \int_0^t X_s^2 ds$ is a martingale, since then by uniqueness of the Doob-Meyer decomposition we must have that $\langle I_t(X) \rangle = \int_0^t X_s^2 ds$. But note that (4) is equivalent to

$$\begin{aligned}
\mathbb{E}[I_t^2(X) - I_s^2(X)|\mathcal{F}_s] = \mathbb{E}[I_t^2(X)|\mathcal{F}_s] - I_s^2(X) &= \mathbb{E}[\int_s^t X_u^2 du|\mathcal{F}_s] \\
&= \mathbb{E}[\int_0^t X_u^2 du|\mathcal{F}_s] - \mathbb{E}[\int_0^s X_u^2 du|\mathcal{F}_s] \\
&= \mathbb{E}[\int_0^t X_u^2 du|\mathcal{F}_s] - \int_0^s X_u^2 du.
\end{aligned}$$

Namely,

$$\mathbb{E}[I_t^2(X)|\mathcal{F}_s] - \mathbb{E}[\int_0^t X_u^2 du|\mathcal{F}_s] = I_s^2(X) - \int_0^s X_u^2 du,$$

namely, $I_t^2(X) - \int_0^t X_s^2 ds$ is indeed a martingale. □

3 Additional reading materials

- Karatzas and Shreve [1].
- Øksendal [2], Chapter III.

References

[1] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, Springer, 1991.

[2] B. Øksendal, *Stochastic differential equations*, Springer, 1991.

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