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Functional Law of Large Numbers. Construction of the Wiener Measure

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1 Additional technical results on weak convergence

Given two metric spaces S_1, S_2 and a measurable function $f : S_1 \to S_2$, suppose S_1 is equipped with some probability measure $\mathbb{P}$. This induces a probability measure on S_2 which is denoted by $\mathbb{P}f^{-1}$ and is defined by $\mathbb{P}f^{-1}(A) = \mathbb{P}(f^{-1}(A))$ for every measurable set $A \subset S_2$. Then for any random variable $X : S_2 \to \mathbb{R}$, its expectation $\mathbb{E}_{\mathbb{P}f^{-1}}[X]$ is equal to $\mathbb{E}_{\mathbb{P}}[X(f)]$. (Convince yourself that this is the case by looking at the special case when f is a simple function).

Theorem 1 (Mapping Theorem)**.** *Suppose $\mathbb{P}_n \Rightarrow \mathbb{P}$ for a sequence of probability measures $\mathbb{P}, \mathbb{P}_n$ on S_1 and suppose $f : S_1 \to S_2$ is continuous. Then $\mathbb{P}_n f^{-1} \Rightarrow \mathbb{P}f^{-1}$ on S_2.*

Proof. We use Portmentau theorem, in particular weak convergence characterization using bounded continuous functions. Thus let $g : S_2 \to \mathbb{R}$ be any bounded continuous function. Since it is continuous, it is also measurable, thus it is also a random variable defined on $(S_2, \mathcal{B}_2)$, where $\mathcal{B}_2$ is the Borel σ-field on S_2. We have,

$$\mathbb{E}_{\mathbb{P}_n f^{-1}}[g] = \mathbb{E}_{\mathbb{P}_n}[g(f)].$$

Since g is a bounded continuous, then the composition is also bounded continuous. Therefore, by Portmanteau theorem

$$\mathbb{E}_{\mathbb{P}_n}[g(f)] \to \mathbb{E}_{\mathbb{P}}[g(f)] = \mathbb{E}_{\mathbb{P}f^{-1}}[g]$$

$\square$

Definition 1. *A sequence of probability measures* $\mathbb{P}_n$ *on metric space* S *is defined to be* tight *if for every* $\epsilon > 0$ *there exists* n_0 *and a compact set* $K \subset S$*, such that* $\mathbb{P}_n(K) > 1 - \epsilon$ *for all* $n > n_0$.

Theorem 2 (Prohorov's Theorem)**.** *Suppose sequence* $\mathbb{P}_n$ *is tight. Then it contains a weakly convergent subsequence* $\mathbb{P}_{n(k)} \Rightarrow \mathbb{P}$.

The converse of this theorem is also true, but we will not need this. We do not prove Prohorov's Theorem. The proof can be found in [1]. Recall that Arzela-Ascoli Theorem provides a characterization of compact sets in $C[0, T]$. We can use it now for characterization of tightness.

Proposition 1. *Suppose a sequence of measures* $\mathbb{P}_n$ *on* $C[0, T]$ *satisfies the following conditions:*

(i) There exists $a \geq 0$ *such that* $\lim_n \mathbb{P}_n(|x(0)| \geq a) = 0$.

(ii) For each $\epsilon > 0$, $\lim_{\delta \to 0} \limsup_n \mathbb{P}_n(\{x : w_x(\delta) > \epsilon\}) = 0$.

Then the sequence $\mathbb{P}_n$ *is tight.*

Proof. Fix $\epsilon > 0$. From (i) we can find $\hat{a}$ and n_0 large enough so that $\mathbb{P}_n(|x(0)| > \hat{a}) < \epsilon$ for all $n > n_0$. For every $m \leq n_0$ we can also find a_m large enough so that $\mathbb{P}_m(|x(0)| > a_m) < \epsilon$. Take $a = \max(\hat{a}, a_m)$. Then $\mathbb{P}_n(|x(0)| > a) < \epsilon$ for all n. Let $B = \{x : |x(0)| \leq a\}$. We just showed $\mathbb{P}_n(B) \geq 1 - \epsilon$.

Similarly, for every $k > 0$ we can find $\hat{\delta}_k$ and n_k such that $\mathbb{P}_n(w_x(\hat{\delta}_k) > \epsilon/2^k) < \epsilon/2^k$ for all $n > n_k$. For every fixed $n \leq n_k$ we can find a small enough $\delta_n > 0$ such that $\mathbb{P}_n(w_x(\delta_n) > \epsilon/2^k) < \epsilon/2^k$ since by uniform continuity of x we have $\cap_{\delta > 0}\{w_x(\delta) > \epsilon/2^k\} = \emptyset$ a.s. Let $\delta_k = \min(\hat{\delta}_k, \min_{n \leq n_k} \delta_n)$. Let $B_k = \{x : w_x(\delta_k) \leq \epsilon\}$. Since $\mathbb{P}_n(B_k^c) < \epsilon/2^k$ then $\mathbb{P}_n(\cup_k B_k^c) < \epsilon$ and $\mathbb{P}_n(\cap_k B_k) \geq 1 - \epsilon$, for all n. Therefore $\mathbb{P}_n(B \cap \cap_k B_k) \geq 1 - 2\epsilon$ for all n. Then set $K = B \cap \cap_k B_k$ is closed (check) and satisfies the conditions of Arzela-Ascoli Theorem. Therefore it is compact. $\square$

2 Functional Strong Law of Large Numbers (FSLLN)

We are about to establish two very important limit results in the theory of stochastic processes. In probability theory two cornerstone theorems are (Weak or Strong) Law of Large Numbers and Central Limit Theorem. These theorems have direct analogue in the theory of stochastic processes as Functional Strong Law of Large Numbers (FSLLN) and Functional Central Limit Theorem (FCLT) also known as Donsker Theorem. The second theorem contains in it the fact that Wiener Measure exists.

We first describe the setup. Consider a sequence of i.i.d. random variables $X_1, X_2, \ldots, X_n, \ldots$. We assume that $\mathbb{E}[X_1] = 0, \mathbb{E}[X_1^2] = \sigma^2$. We can view each realization of an infinite sequence $(X_n(\omega))$ as a sample in a product space $\mathbb{R}^\infty$ equipped with product type σ-field $\mathcal{F}$ and probability measure $\mathbb{P}_{\text{i.i.d.}}$, induced by the probability distribution of X_1.

Define $S_n = \sum_{1 \le k \le n} X_k$. Fix an interval $[0, T]$ and for each $n \ge 1$ and $t \in [0, T]$ consider the following function

$$N_n(t) = \frac{S_{\lfloor nt \rfloor}(\omega)}{n} + (nt - \lfloor nt \rfloor)\frac{X_{\lfloor nt \rfloor + 1}(\omega)}{n}. \tag{1}$$

This is a piece-wise linear continuous function in $C[0, T]$.

Theorem 3 (**Functional Strong Law of Large Numbers (FSLLN)**). *Given an i.i.d. sequence $(X_n), n \ge 1$ with $\mathbb{E}[X_1] = 0, \mathbb{E}[|X_1|] < \infty$, for every $T > 0$, the sequence of functions $N_n : [0, T] \to \mathbb{R}$ converges to zero almost surely. Namely*

$$\mathbb{P}(\|N_n(\omega)\|_T \to 0) = \mathbb{P}(\sup_{0 \le t \le T} |N_n(t, \omega)| \to 0) = 1$$

As we see, just as SLLN, the FSLLN holds without any assumptions on the variance of X_1, that is even if $\sigma = \infty$.

Here is another way to state FSLLN. We may consider functions N_n defined on entire $[0, \infty)$ using the same defining identity (1). Recall that sets $[0, T]$ are compact in $\mathbb{R}$. An equivalent way of saying FSLLN is N_n converges to zero almost surely uniformly on compact sets.

Proof. Fix $\epsilon > 0$ and $T > 0$. By SLLN we have that for almost all realizations ω of an sequence $X_1(\omega), X_2(\omega), \ldots$, there exists $n_0(\omega)$ such that for all $n > n_0(\omega)$,

$$\Big|\frac{S_n(\omega)}{n}\Big| < \frac{\epsilon}{T}$$

We let $M(\omega) = \max_{1 \le m \le n_0(\omega)} S_m(\omega)$. We claim that for $n > M(\omega)/\epsilon$, there holds

$$\sup_{0 \le t \le T} |N_n(t)| < \epsilon.$$

We consider two cases. Suppose $t \in [0, T]$ is such that $nt > n_0(\omega)$. Then

$$N_n(t) \le \max\Big(\frac{S_{\lfloor nt \rfloor}(\omega)}{n}, \frac{S_{\lfloor nt \rfloor + 1}(\omega)}{n}\Big).$$

We have

$$\frac{S_{\lfloor nt \rfloor}(\omega)}{n} = \frac{S_{\lfloor nt \rfloor}(\omega)}{\lfloor nt \rfloor}\frac{\lfloor nt \rfloor}{n} \le \frac{\epsilon}{T}t \le \epsilon.$$

Using a similar bound on $S_{\lfloor nt \rfloor+1}(\omega)$, we obtain

$$\frac{N_n(t)}{n} \le \epsilon.$$

Suppose now t is such that $nt \le n_0(\omega)$. Then

$$|N_n(t)| \le \frac{M(\omega)}{n} < \epsilon,$$

since, by our choice $n > M(\omega)/\epsilon$. We conclude $\sup_{0\le t\le T}|N_n(t)| < \epsilon$ for all $n > M(\omega)/\epsilon$. This concludes the proof. □

3 Weiner measure

FSSLN was a simpler functional limit theorem. Here we consider instead a "Gaussian" scaling of a random walk S_n and establish existence of the Weiner measure (Brownian motion) as well as FCLT. Thus suppose we have a sequence of i.i.d. random variables $X_1, \dots, X_n$ with mean zero but finite variance $\sigma^2 < \infty$. Instead of function (1) consider the following function

$$N_n(t,\omega) = \frac{S_{\lfloor nt \rfloor}(\omega)}{\sigma\sqrt{n}} + (nt - \lfloor nt \rfloor)\frac{X_{\lfloor nt \rfloor+1}(\omega)}{\sigma\sqrt{n}}, \qquad n \ge 1,\ t \in [0,T]. \quad (2)$$

This is again a piece-wise linear continuous function. Then for each n we obtain a mapping

$$\psi_n : \mathbb{R}^\infty \to C[0,T].$$

Of course, for each n, the mapping ψ_n depends only on the first $nT + 1$ coordinates of samples in $\mathbb{R}^\infty$.

Lemma 1. *Each mapping ψ_n is measurable.*

Proof. Here is where it helps to know that Kolmogorov field is identical to Borel field on $C[0,T]$, that is Theorem 1.4 from the previous lecture. Indeed, now it suffices to show that that $\psi_n^{-1}(A)$ is measurable for each set A of the form

$A = \pi_t^{-1}(-\infty, y]$, as these sets generate Kolmogorov/Borel σ-field. Each set $\psi_n^{-1}(\pi_t^{-1}(-\infty, y])$ is the set of all realizations $N_n(\omega)$ such that

$$N_n(t,\omega) = \frac{\sum_{1\le k\le m} X_k(\omega)}{\sigma\sqrt{n}} + (nt - m)X_{m+1}(\omega) \le y.$$

where $m = \lfloor nt \rfloor$. This defines a measurable subset of $\mathbb{R}^{m+1} \subset \mathbb{R}^\infty$. One way to see this is to observe that the function $f : \mathbb{R}^{m+1} \to \mathbb{R}$ defined by $f(x_1, \ldots, x_m) = \frac{\sum_{1\le k\le m} x_k}{\sigma\sqrt{n}} + (nt - m)x_{m+1}$ is continuous and therefore is measurable. We conclude that ψ_n is measurable for each n. $\square$

Thus each ψ_n induces a probability measure on $C[0, T]$, which we denote by $\mathbb{P}_n$. This probability measure is defined by

$$\mathbb{P}_n(A) = \mathbb{P}_{\text{i.i.d.}}(\psi_n^{-1}(A)) = \mathbb{P}_{\text{i.i.d.}}(N_n(\omega) \in A).$$

We now establish the principal result of this lecture – existence of Weiner measure, namely, the existence of a Brownian motion.

Theorem 4 (Existence of Wiener measure)**.** *A sequence of measures $\mathbb{P}_n$ has a weak limit $\mathbb{P}^*$ which satisfies the property of Wiener measure on $C[0, T]$.*

The proof of this fact is quite involved and we give only its scheme, skipping some technical results. First let us outline the main steps in the proof. In the previous lecture we considered projection mappings $\mathbb{P}_t : C[0, T] \to \mathbb{R}$. Similarly, for any collection $0 \le t_1 < \cdots < t_k$ we can consider $\pi_{t_1,\ldots,t_k}(x) = (x(t_1), \ldots, x(t_k)) \in \mathbb{R}^k$.

1. We first show that the sequence of measures $\mathbb{P}_n$ on $C[0, T]$ is tight. We use this to argue that there exists a subsequence $\mathbb{P}_{n(k)}$ which converges to some measure π^*.

2. We show that π^* satisfies the properties of Wiener measures. For this purposes we look at the projected measures $\pi_{t_1,\ldots,t_k}(\pi^*)$ on $\mathbb{R}^k$ and show that these give a joint Gaussian distribution, the kind arising in a Brownian motion (that is the joint distribution of $(B(t_1), B(t_2), \ldots, B(t_k))$). At this point the existence of Wiener measure is established.

3. We then show that in fact the weak convergence $\pi_n^* \Rightarrow \pi$ holds.

Proof sketch. We begin with the following technical and quite delicate result about random walks.

Lemma 2. *The following identity holds for random walks $S_n = X_1 + \cdots + X_n$:*

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} \lambda^2 \mathbb{P}(\max_{k\le n} |S_k| \ge \lambda\sigma\sqrt{n}) = 0. \tag{3}$$

Note, that this is indeed a very subtle result. We could try to use sub-martingale inequality, since S_k is sub-martingale. It will give

$$\mathbb{P}(\max_{k\le n} |S_k| \ge \lambda\sigma\sqrt{n}) \le \frac{\mathbb{E}[S_n^2]}{\lambda^2\sigma^2 n} = \frac{1}{\lambda^2}.$$

So by taking a product with λ^2 we do not obtain convergence to zero. On the other hand note that if the random variables have a finite fourth moment $\mathbb{E}[X_n^4] < \infty$, then the result follows from the sub-martingale inequality by considering S_n^4 in place of S_n^2 (exercise). The proof of this lemma is based on the following fact:

Proposition 2 (Etemadi's Inequality,[1])**.** *For every $\alpha > 0$*

$$\mathbb{P}(\max_{k\le n} |S_k| \ge 3\alpha) \le 3 \max_{k\le n} \mathbb{P}(|S_k| \ge \alpha)$$

Proof. Let B_k be the event $|S_k| \ge 3\alpha, |S_j| < 3\alpha, j < k$. Then

$$\begin{aligned}
\mathbb{P}(\max_{k\le n} |S_k| \ge 3\alpha) &\le \mathbb{P}(|S_n| \ge \alpha) + \sum_{k\le n} \mathbb{P}(B_k \cap |S_n| < \alpha) \\
&\le \mathbb{P}(|S_n| \ge \alpha) + \sum_{k\le n} \mathbb{P}(B_k \cap |S_n - S_k| > 2\alpha) \\
&= \mathbb{P}(|S_n| \ge \alpha) + \sum_{k\le n} \mathbb{P}(B_k)\mathbb{P}(|S_n - S_k| > 2\alpha) \\
&\le \mathbb{P}(|S_n| \ge \alpha) + \max_{k\le n} \mathbb{P}(|S_n - S_k| \ge 2\alpha) \\
&\le \mathbb{P}(|S_n| \ge \alpha) + \max_{k\le n} (\mathbb{P}(|S_n| \ge \alpha) + \mathbb{P}(|S_k| \ge \alpha)) \\
&\le 3 \max_{k\le n} \mathbb{P}(|S_k| \ge \alpha).
\end{aligned}$$

□

Now we can prove Lemma 2.

Proof of Lemma 2. Applying Etemadi's Inequality

$$\mathbb{P}(\max_{k\le n} |S_k| \ge \lambda\sigma\sqrt{n}) \le 3 \max_{k\le n} \mathbb{P}(|S_k| \ge (1/3)\lambda\sigma\sqrt{n})$$

Fix $\epsilon > 0$. Let Φ denote the cumulative standard normal distribution. Find λ_0 large enough so that $2\lambda^2(1-\Phi(\lambda/3)) < \epsilon/3$ for all $\lambda \geq \lambda_0$. Fix any such λ. By the CLT we can find $n_0 = n_0(\lambda)$ large enough so that

$$\mathbb{P}(|S_n| \geq (1/3)\lambda\sigma\sqrt{n}) \leq 2(1-\Phi(\lambda/3)) + \epsilon/(3\lambda^2)$$

for all $n \geq n_0$, implying $\lambda^2\mathbb{P}(|S_n| \geq (1/3)\lambda\sigma\sqrt{n}) \leq 2\epsilon/3$.

Now fix any $n \geq 27n_0/\epsilon$ and any $k \leq n$. If $k \geq n_0$, then from the derived bound we have

$$\lambda^2\mathbb{P}(|S_k| \geq (1/3)\lambda\sigma\sqrt{n}) \leq \lambda^2\mathbb{P}(|S_k| \geq (1/3)\lambda\sigma\sqrt{k}) \leq 2\epsilon/3.$$

On the other hand, if $k \leq n_0$ then

$$\lambda^2\mathbb{P}(|S_k| \geq (1/3)\lambda\sigma\sqrt{n}) \leq \frac{\lambda^2\mathbb{E}[S_k^2]}{(\lambda^2/9)\sigma^2 n} = \frac{\sigma^2 k}{(1/9)\sigma^2 n} \leq \epsilon/3.$$

We conclude that for all $n \geq 27n_0/\epsilon$,

$$\lambda^2 \max_{1\leq k\leq n} \mathbb{P}(|S_k| \geq (1/3)\lambda\sigma\sqrt{n}) \leq \epsilon/3,$$

from which we obtain

$$\lambda^2 \limsup_n \mathbb{P}(\max_{k\leq n}|S_k| \geq \lambda\sigma\sqrt{n}) \leq \epsilon$$

Since $\epsilon > 0$ was arbitrary, we obtain the result.

□

The next result which we also do not prove says that the property (3) implies tightness of the sequence of measures $\mathbb{P}_n$ on $C[0,T]$.

Lemma 3. *The following convergence holds for every $\epsilon > 0$.*

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}(w_{N_n}(\delta) \geq \epsilon) = 0. \tag{4}$$

As a result the sequence of measures $\mathbb{P}_n$ is tight.

Proof. Observe that

$$w_{N_n}(\delta) \leq \max_{i\leq j\leq nT: j-i\leq\delta n} \frac{|\ {}_{i\leq k\leq j} X_k|}{\sigma\sqrt{n}}.$$

Exercise 1. Use this to finish the proof of the lemma. *Hint:* partition interval $[0,T]$ into length δ intervals and use Lemma 2.

□

Let us now see how Lemma 2 implies tightness. We use characterization given by Proposition 1. First for any positive a, $\mathbb{P}_n(|x(0)| \geq a) = \mathbb{P}(|N_n(0)| > 0) = 0$ since $N_n(0) = 0$. Now $\mathbb{P}_n(w_x(\delta) > \epsilon) = \mathbb{P}(w_{N_n}(\delta) > \epsilon)$. From the first part of the lemma we know that the double convergence (4) holds. This means that condition (ii) of Proposition 1 holds as well.

We now return to the construction of Wiener measure. Lemma 3 implies that the sequence of probability measures $\mathbb{P}_n$ on $C[0,T]$ is tight. Therefore, by Prohorov's Theorem, it contains a weakly convergent subsequence $\mathbb{P}_{n(k)} \Rightarrow \mathbb{P}^*$.

Proposition 3. $\mathbb{P}^*$ *satisfies the property of the Wiener measure.*

Proof. Since $\mathbb{P}^*$ is defined on the space of continuous functions, then continuity of every sample is immediate. We need to establish independence of increments and the fact that increments are stationary Gaussian. Thus we fix $0 \leq t_1 < \cdots < t_k$ and $y_1, \ldots, y_k \in \mathbb{R}^k$. To preserve the continuity, we still denote elements of $C[0,T]$ by $x, x(t)$ or $x(\omega, t)$ whereas before we used notations $\omega, B(\omega), B(t,\omega)$.

Consider the random vector $\pi_{t_1}(N_n) = N_n(t_1)$. This is simply the random variable

$$\frac{S_{\lfloor nt_1 \rfloor}(\omega)}{\sigma\sqrt{n}} + (nt_1 - \lfloor nt_1 \rfloor)\frac{X_{\lfloor nt_1 \rfloor + 1}(\omega)}{\sigma\sqrt{n}}$$

The second term in the sum converges to zero in probability. The first term we rewrite as

$$\frac{S_{\lfloor nt_1 \rfloor}(\omega)}{\sigma\lfloor\sqrt{nt_1}\rfloor}\frac{\lfloor\sqrt{nt_1}\rfloor}{\sqrt{n}}.$$

and by CLT it converges to a normal $N(0, t_1)$ distribution. Similarly, consider

$$N_n(t_2) - N_n(t_1) = \frac{\sum_{nt_1 < m \leq nt_2} X_m(\omega)}{\sigma\sqrt{n}} + (nt_2 - \lfloor nt_2 \rfloor)\frac{X_{\lfloor nt_2 \rfloor + 1}(\omega)}{\sigma\sqrt{n}} - (nt_1 - \lfloor nt_1 \rfloor)\frac{X_{\lfloor nt_1 \rfloor + 1}(\omega)}{\sigma\sqrt{n}}$$

Again by CLT we see that it converges to normal $N(0, t_2 - t_1)$ distribution. Moreover, the joint distribution of $(N_n(t_1), N_n(t_2) - N_n(t_1))$ converges to a joint distribution of two independent normals with zero mean and variances $t_1, t_2 - t_1$. Namely, the $(N_n(t_1), N_n(t_2))$ converges in distribution to

$(Z_1, Z_1 + Z_2)$, where Z_1, Z_2 are independent normal random variables with zero mean and variances $t_1, t_2 - t_1$.

By a similar token, we see that the distribution of the random vector $\pi_{t_1,\ldots,t_k}(N_n) = (N_n(t_1), N_n(t_2), \ldots, N_n(t_k))$ converges in distribution to $(Z_1, Z_1 + Z_2, \ldots, Z_1 + \cdots + Z_k)$ where $Z_j, 1 \leq j \leq k$ are independent zero mean normal random variables with variances $t_1, t_2 - t_1, \ldots, t_k - t_{k-1}$.

On the other hand the distribution of $\pi_{t_1,\ldots,t_k}(N_n)$ is

$$\mathbb{P}_n \pi^{-1}_{t_1,\ldots,t_k} = \mathbb{P}_{\text{i.i.d.}} \psi_n^{-1} \pi^{-1}_{t_1,\ldots,t_k}.$$

Since $\mathbb{P}_{n(k)} \Rightarrow \mathbb{P}^*$ and π is continuous, then, applying mapping theorem (Theorem 1) we conclude that $\mathbb{P}_{n(k)} \pi^{-1}_{t_1,\ldots,t_k} \Rightarrow \mathbb{P}^* \pi^{-1}_{t_1,\ldots,t_k}$. Combining, these two facts, we conclude that the probability measure $\mathbb{P}^* \pi^{-1}_{t_1,\ldots,t_k}$ is the probability measure of $(Z_1, Z_1 + Z_2, \ldots, Z_1 + \cdots + Z_k)$ (where again Z_j are independent normal, etc ...). What does this mean? This means that when we select $x \in C[0,T]$ according to the probability measure $\mathbb{P}^*$ and look at its projection $\pi_{t_1,\ldots,t_k}(x) = (x(t_1), \ldots, x(t_k))$, the probability distribution of this random vector is the distribution of $(Z_1, Z_1 + Z_2, \ldots, Z_1 + \cdots + Z_k)$. This means that x has independent increments with zero mean Gaussian distribution and variances $t_1, t_2 - t_1, \ldots, t_k - t_{k-1}$. This is precisely the property we needed to establish for $\mathbb{P}^*$ in order to argue that it is indeed Wiener measure.

This concludes the proof of Proposition 3 – the fact that $\mathbb{P}^*$ is the Weiner measure. $\square$

We also need to show that the convergence $\mathbb{P}_n \Rightarrow \mathbb{P}$ holds. For this purpose we will show that $\mathbb{P}^*$ is unique. In this case the convergence holds. Indeed, suppose otherwise, there exists a subsequence $n(k)$ such that $\mathbb{P}_{n(k)} \not\Rightarrow \mathbb{P}^*$. Then we can find a bounded continuous r.v. X, such that $\mathbb{E}_{\mathbb{P}_{n(k)}} X \not\Rightarrow \mathbb{E}_{\mathbb{P}^*} X$. Then we can find ϵ_0 and a subsequence $n(k_i)$ of $n(k)$ such that $|\mathbb{E}_{\mathbb{P}_{n(k_i)}} X - \mathbb{E}_{\mathbb{P}^*} X| \geq \epsilon_0$ for all i. By tightness we can find a further subsequence $n(k_{i_j})$ of $n(k_i)$ which converges weakly to some limiting probability measure $\tilde{\mathbb{P}}$. But we have seen that every such weak limit has to be a Wiener measure which is unique. Namely $\tilde{\mathbb{P}} = \mathbb{P}^*$. This is a contradiction since $|\mathbb{E}_{\mathbb{P}_{n(k_{i_j})}} X - \mathbb{E}_{\mathbb{P}^*} X| \geq \epsilon_0$.

It remains to show the uniqueness of Wiener measure. This follows again from the fact that the Kolmogorov σ-field coincides with the Borel σ-field on $C[0,T]$. But the properties of Wiener measure (independent increments with variances given by the length of the time increments) uniquely define probability on generating sets obtained via projections $\pi_{t_1,\ldots,t_k} : C[0,T] \to \mathbb{R}^k$. This concludes the proof of uniqueness. $\square$

4 Applications

Theorem 4 has applications beyond the existence of Wiener measure. Here is one of them.

Theorem 5. *The following convergence holds*

$$\frac{\max_{1\le k\le n} S_k}{\sigma\sqrt{n}} \Rightarrow \sup_{0\le t\le T} B(t) \tag{5}$$

where B is the standard Brownian motion. As a result, for every y

$$\lim_n \mathbb{P}(\frac{\max_{1\le k\le n} S_k}{\sigma\sqrt{n}} \ge y) = 2(1-\Phi(y)), \tag{6}$$

where Φ is standard normal distribution.

Proof. The function $g(x) = \sup_{0\le t\le T} x(t)$ is a continuous function on $C[0,T]$ (check this). Since by Theorem 4, $\mathbb{P}_n \Rightarrow \mathbb{P}^*$ then, by Mapping Theorem, $g(N_n) \Rightarrow g(B)$, where B is a standard Brownian motion – random sample from the Wiener measure $\mathbb{P}^*$. But $g(N_n) = \sup_{0\le t\le T} N_n(t) = \frac{\max_{1\le k\le n} S_k}{\sigma\sqrt{n}}$. We conclude that (5) holds. To prove the second part we note that the set $A = \{x \in C[0,T] : \sup_{0\le t\le T} x(t) = y\}$ has $\mathbb{P}^*(A) = 0$ – recall that the maximum $\sup_{0\le t\le T} B(t)$ of a Brownian motion has density. Also note that A is the boundary of the set $\{x : \sup_{0\le t\le T} x(t) \le y\}$. Therefore by Portmentau theorem, (6) holds.

□

5 Additional reading materials

- Billingsley [1] Chapter 2, Section 8.

References

[1] P. Billingsley, *Convergence of probability measures*, Wiley-Interscience publication, 1999.

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