Applications of the large deviation technique

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1 Safety capital for an insurance company

Consider some insurance company which needs to decide on the amount of capital S_0 it needs to hold to avoid the cash flow issue. Suppose the insurance premium per month is a fixed (non-random) quantity $C > 0$. Suppose the claims are i.i.d. random variable $A_N \geq 0$ for the time $N = 1, 2, \cdots$. Then the capital at time N is $S_N = S_0 + \sum_{n=1}^{N}(C - A_n)$. The company wishes to avoid the situation where the cash flow S_N is negative. Thus it needs to decide on the capital S_0 so that $\mathbb{P}(\exists N, S_N \leq 0)$ is small. Obviously this involves a tradeoff between the "smallness" and the amount S_0. Let us assume that upper bound $\delta = 0.001$, namely 0.1% is acceptable (in fact this is pretty close to the banking regulation standards). We have

$$\mathbb{P}(\exists N, S_N \leq 0) = \mathbb{P}(\min_N S_0 + \sum_{n=1}^{N}(C - A_n) \leq 0)$$
$$= \mathbb{P}(\max_N \sum_{n=1}^{N}(A_n - C) \geq S_0)$$

If $\mathbb{E}[A_1] \geq C$, we have $\mathbb{P}(\max_N \sum_{n=1}^{N}(A_n - C) \geq S_0) = 1$. Thus, the interesting case is $\mathbb{E}[A_1] < C$ (negative drift), and the goal is to determine the starting capital S_0 such that

$$\mathbb{P}(\max_N \sum_{n=1}^{N}(A_n - C) \geq S_0) \leq \delta.$$

2 Buffer overflow in a queueing system

The following model is a variant of a classical so called GI/GI/1 queueing system. In application to communication systems this queueing system consists of a single server, which processes some $C > 0$ number of communication packets per unit of time. Here C is a fixed deterministic constant. Let A_n be the random number packets arriving at time n, and Q_n be the queue length at time n (asume Q_0=0). By recursion, we have that

$$\begin{aligned} Q_N &= \max(Q_{N-1} + A_N - C, 0) \\ &= \max(Q_{N-2} + A_{N-1} + A_N - 2C, A_N - C, 0) \\ &= \max_{1\leq n\leq N-1} (\sum_{k=1}^{n} (A_{N-k} - C), 0) \end{aligned}$$

Notice, that in distributional sense we have

$$Q_N = \max\left(\max_{1\leq n\leq N-1}\left(\sum_{k=1}^{n}(A_{N-k} - C)\right), 0\right)$$

In steady state, i.e. $N = \infty$, we have

$$Q_\infty = \max\left(\max_{n\geq 1}\sum_{k=1}^{n}(A_k - C), 0\right)$$

Our goal is to design the size of the queue length storage (buffer) B, so that the likelihood that the number of packets in the queue exceeds B is small. In communication application this is important since every packet not fitting into the buffer is dropped. Thus the goal is to find buffer size $B > 0$ such that

$$\mathbb{P}(Q_\infty \geq B) \leq \delta \Rightarrow \mathbb{P}(\max_{n\geq 1}\sum_{k=1}^{n}(A_k - C) \geq B) \leq \delta$$

If $\mathbb{E}[A_1] \geq C$, we have $\mathbb{P}(Q_\infty \geq B) = 1$. So the interesting case is $\mathbb{E}[A_1] < C$ (negative drift).

3 Buffer overflow probability

We see that in both situations we need to estimate

$$\mathbb{P}(\max_{n\geq 1}\sum_{k=1}^{n}(A_k - C) \geq B).$$

We will do this asymptoticlly as $B \to \infty$.

Theorem 1. *Given an i.i.d. sequence* $A_n \geq 0$ *for* $n \geq 1$ *and* $C > \mathbb{E}[A_1]$. *Suppose*

$$M(\theta) = \mathbb{E}[\exp(\theta A)] < \infty, \text{ for some } \theta \in [0, \theta_0).$$

Then

$$\lim_{B\to\infty} \frac{1}{B} \log \mathbb{P}(\max_{n\geq 1} \sum_{k=1}^{n} (A_k - C) \geq B) = -\sup\{\theta > 0 : M(\theta) < \exp(\theta C)\}$$

Observe that since A_n is non-negative, the MGF $\mathbb{E}[\exp(\theta A_n)]$ is finite for $\theta < 0$. Thus it is finite in an interval containing $\theta = 0$, and applying the result of Lecture 2 we can take the derivative of MGF. Then

$$\frac{d}{d\theta} M(\theta)\Big|_{\theta=0} = \mathbb{E}[A], \quad \frac{d}{d\theta} \exp(\theta C)\Big|_{\theta=0} = C$$

Since $\mathbb{E}[A_n] < C$, then there exists small enough θ so that $M(\theta) < \exp(\theta C)$,

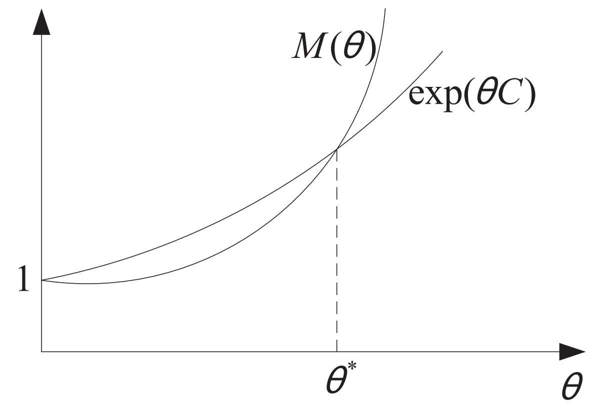

Figure 1: Illustration for the existance of θ such that $M(\theta) < \exp(\theta C)$

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and thus the set of $\theta > 0$ for which this is the case is non-empty. (see Figure 1). The theorem says that roughly speaking

$$\mathbb{P}(\max_{n} \sum_{k=1}^{n} (A_k - C) \geq B) \sim \exp(-\theta^* B),$$

when B is large. Thus given δ select B such that $\exp(-\theta^* B) \leq \delta$, and we can set $B = \frac{1}{\theta^*} \log \frac{1}{\delta}$.

Example. Let A be a random variable uniformly distributed in $[0, a]$ and $C = 2$. Then, the moment generating function of A is

$$M(\theta) = \int_0^a \exp(\theta t)a^{-1}\,\mathrm{d}t = \frac{\exp(\theta a) - 1}{\theta a}$$

Then

$$\sup\{\theta > 0 : M(\theta) \leq \exp(\theta C)\} = \sup\{\theta > 0 : \frac{\exp(\theta a) - 1}{\theta a} \leq \exp(2\theta)\}$$

Case 1: $a = 3$, we have $\theta^* = \sup\{\theta > 0 : \exp(3\theta) - 1 \leq 3\theta \exp(2\theta)\}$, i.e. $\theta^* = 1.54078$.
Case 2: $a = 4$, we have that $\{\theta > 0 : \exp(3\theta) - 1 \leq 3\theta \exp(2\theta)\} = \emptyset$ since $\mathbb{E}[A] = 2 = C$.
Case 3: $a = 2$, we have that $\{\theta > 0 : \exp(3\theta) - 1 \leq 3\theta \exp(2\theta)\} = \mathbb{R}_+$ and thus $\theta^* = \infty$, which implies that $\mathbb{P}(\max_n \sum_{k=1}^n (A_k - C) \geq B) = 0$ by theorem 1.

Proof of Theorem 1. We will first prove an upper bound and then a lower bound. Combining them yields the result. For the upper bound, we have that

$$\begin{aligned}
\mathbb{P}(\max_n \sum_{k=1}^n (A_k - C) \geq B) &\leq \sum_{n=1}^{\infty} \mathbb{P}(\sum_{k=1}^n (A_k - C) \geq B) \\
&= \sum_{n=1}^{\infty} \mathbb{P}(\frac{1}{n}\sum_{k=1}^n A_k \geq C + \frac{B}{n}) \\
&\leq \sum_{n=1}^{\infty} \exp(-n(\theta(C + \frac{B}{n}) - \log M(\theta)))\ (\theta > 0) \\
&= \exp(-\theta B) \sum_{n \geq 1} \exp(-n(\theta C - \log M(\theta)))
\end{aligned}$$

Fix any θ such that $\theta C \geq \log M(\theta)$, the inequality above gives

$$\begin{aligned}
&\leq \exp(-\theta B) \sum_{n \geq 0} \exp(-n(\theta C - \log M(\theta))) \\
&= \exp(-\theta B)[1 - \exp(-(\theta C - \log M(\theta)))]^{-1}
\end{aligned}$$

Simplification of the inequality above gives

$$\frac{1}{B} \log \mathbb{P}(\max_n \sum_{k=1}^n (A_k - C) \geq B) \leq -\theta + \frac{1}{B} \log([1 - \exp(-(\theta C - \log M(\theta)))]^{-1})$$

$$\Rightarrow \limsup_{B \to \infty} \frac{1}{B} \log \mathbb{P}(\max_n \sum_{k=1}^n (A_k - C) \geq B) \leq -\theta \text{ for } \forall\theta : M(\theta) < \exp(\theta C)$$

Next, we will derive the lower bound.

$$\mathbb{P}(\max_n \sum_{k=1}^{n}(A_k - C) \geq B) \geq \mathbb{P}(\sum_{k=1}^{n}(A_k - C) \geq B), \forall n$$

Fix a $t > 0$, then

$$\begin{aligned}\mathbb{P}(\max_n \sum_{k=1}^{n}(A_k - C) \geq B) &\geq \mathbb{P}(\sum_{k=1}^{\lceil Bt \rceil}(A_k - C) \geq B) \\ &\geq \mathbb{P}\left(\sum_{k=1}^{\lceil Bt \rceil}(A_k - C) \geq \frac{\lceil Bt \rceil}{t}\right)\end{aligned}$$

Then, we have

$$\begin{aligned}&\liminf_B \frac{1}{B} \log \mathbb{P}(\max_n \sum_{k=1}^{n}(A_k - C) \geq B) \\ &\geq \liminf_B \frac{1}{B} \log \mathbb{P}\left(\sum_{k=1}^{\lceil Bt \rceil}(A_k - C) \geq \frac{\lceil Bt \rceil}{t}\right) \\ &= \liminf_B \frac{t}{\lceil Bt \rceil} \log \mathbb{P}\left(\sum_{k=1}^{\lceil Bt \rceil}(A_k - C) \geq \frac{\lceil Bt \rceil}{t}\right) \\ &= t \liminf_n \frac{1}{n} \log \mathbb{P}(\sum_{k=1}^{n}(A_k - C) \geq \frac{n}{t}) \\ &= t \liminf_n \frac{1}{n} \log \mathbb{P}(\frac{1}{n}\sum_{k=1}^{n} A_k \geq C + \frac{1}{t}) \\ &\geq -t \inf_{x > C + \frac{1}{t}} I(x) \quad \text{(by Cramer's theorem.)} \\ &\geq -\inf_{t>0} t \inf_{x > C + \frac{1}{t}} I(x) \quad \text{(since we can choose an arbitrary positive } t. \text{)} \qquad (1)\end{aligned}$$

We claim that

$$-\inf_{t>0} t \inf_{x > C + \frac{1}{t}} I(x) = -\inf_{t>0} tI(C + \frac{1}{t})$$

Indeed, let $x^* = \inf x : I(x) = \infty$ (possibly $x^* = \infty$). If $x^* \leq C$, then $I(C + \frac{1}{t}) = \infty$. Suppose $C < x^*$. If t is such that $C + \frac{1}{t} \geq x^*$, then $\inf_{x > C + \frac{1}{t}} I(x) = \infty$. Therefore it does not make sense to consider such t. Now for $c + \frac{1}{t} < x^*$, we have I is convex non-decreasing and finite on $[E[A_1], x^*)$. Therefore it is continuous on $[E[A_1], x^*)$, which gives that

$$\inf_{x > C + \frac{1}{t}} I(x) = I(C + \frac{1}{t})$$

and the claim follows. Thus, we obtain

$$\liminf_{B \to \infty} \frac{1}{B} \log \mathbb{P}(\max_n \sum_{k=1}^{n} (A_k - C) \geq B) \geq - \inf_{t>0} tI(C + \frac{1}{t})$$

Exercise in HW 2 shows that $\sup\{\theta > 0 : M(\theta) < \exp(C\theta)\} = \inf_{t>0} tI(C + \frac{1}{t})$.

$\square$