MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Brownian motion. Introduction

Content.

1. A heuristic construction of a Brownian motion from a random walk.
2. Definition and basic properties of a Brownian motion.

1 Historical notes

- 1765 Jan Ingenhousz observations of carbon dust in alcohol.
- 1828 Robert Brown observed that "pollen grains suspended in water perform a continual swarming motion".
- 1900 Bachelier's work "The theory of speculation" on using Brownian motion to model stock prices.
- 1905 Einstein and Smoluchovski. Physical interpretation of Brownian motion.
- 1920's Wiener concise mathematical description.

2 Construction of a Brownian motion from a random walk

The developments in this lecture follow closely the book by Resnick [3].

In this section we provide a heuristic "construction" of a Brownian motion from a random walk. The derivation below is not a proof. We will provide a rigorous construction of a Brownian motion when we study the weak convergence theory.

The large deviations theory predicts exponential decay of probabilities $\mathbb{P}(\sum_{1\le i\le n} X_n > an)$ when $a > \mu = \mathbb{E}[X_1]$ and $\mathbb{E}[e^{\theta X_1}]$ are finite. Naturally, the decay will be slower the closer a is to μ. We considered only the case when a was a constant. But what if a is a function of n: $a = a_n$? The Central Limit Theorem tells us that the "decay" disappears when $a_n \approx \frac{1}{\sqrt{n}}$. Recall

Theorem 1 (CLT). *Given an i.i.d. sequence* $(X_n, n \geq 1)$ *with* $\mathbb{E}[X_1] = \mu, \mathrm{var}[X_1] = \sigma^2$. *For every constant* a

$$\lim_{n\to\infty} \mathbb{P}(\frac{\sum_{1\leq i\leq n} X_i - \mu n}{\sigma\sqrt{n}} \leq a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

Now let us look at a sequence of partial sums $S_n = \sum_{1\leq i\leq n}(X_i - \mu)$. For simplicity assume $\mu = 0$ so that we look at $S_n = \sum_{1\leq i\leq n} X_i$. Can we say anything about S_n *as a function of* n? In fact, let us make it a function of a real variable $t \in \mathbb{R}_+$ and rescale it by $\sqrt{n}$ as follows. Define $B_n(t) = \frac{\sum_{1\leq i\leq \lfloor nt\rfloor} X_i}{\sqrt{n}}$ for every $t \geq 0$.

Denote by $N(\mu, \sigma^2)$ the distribution function of a normal r.v. with mean μ and variance σ^2.

1. For every fixed $0 \leq s < t$, by CLT we have the distribution of

$$\frac{\sum_{\lfloor ns\rfloor < i \leq \lfloor nt\rfloor} X_i}{\sigma\sqrt{\lfloor nt\rfloor - \lfloor ns\rfloor}}$$

   converging to the standard normal distribution as $n \to \infty$. Ignoring the difference between $\lfloor nt\rfloor - \lfloor ns\rfloor$ and $n(t-s)$ which is at most 1, and writing

$$\frac{\sum_{st < i \leq nt} X_i}{\sigma\sqrt{nt - ns}} = \frac{B_n(t) - B_n(s)}{\sigma\sqrt{t-s}}$$

   we obtain that $B_n(t) - B_n(s)$ converges in distribution to $N(0, \sigma^2(t-s))$.

2. Fix $t_1 < t_2$ and consider $B_n(t_1) = \frac{\sum_{1\leq i\leq nt_1} X_i}{\sqrt{n}}$ and $B_n(t_2) - B_n(t_1) = \frac{\sum_{nt_1 < i \leq nt_2} X_i}{\sqrt{n}}$. The two sums contain different elements of the sequence $X_1, X_2, \ldots$. Since the sequence is i.i.d. $B_n(t_1)$ and $B_n(t_2) - B_n(t_1)$ are independent. Namely for every x_1, x_2

$$\mathbb{P}(B_n(t_1) \leq x_1, B_n(t_2) - B_n(t_1) \leq x_2) = \mathbb{P}(B_n(t_1) \leq x_1)\mathbb{P}(B_n(t_2) - B_n(t_1) \leq x_2).$$

   This generalizes to any finite collection of *increments* $B_n(t_1), B_n(t_2) - B_n(t_1), \ldots, B_n(t_k) - B_n(t_{k-1})$. Thus $B_n(t)$ has independent increments.

3. Given a small ϵ, for every t the difference $B_n(t+\epsilon) - B_n(t)$ converges in distribution to $N(0, \sigma^2\epsilon)$. When ϵ is very small this difference is very close to zero with probability approaching 1 as $n \to \infty$. Namely, $B_n(t)$ is increasingly close to a continuous function as $n \to \infty$.

4. $B_n(0) = 0$ by definition.

3 Definition

The Brownian motion is the limit $B(t)$ of $B_n(t)$ as $n \to \infty$. We will delay answering questions about the existence of this limit as well as the sense in which we take this limit (remember we are dealing with processes and not values here) till future lectures and now simply postulate the existence of a process satisfying the properties above. In the theorem below, for every continuous function $\omega \in C[0,\infty)$ we let $B(t,\omega)$ denote $\omega(t)$. This notation is more consistent with a standard convention of denoting Brownian motion by B and its value at time t by $B(t)$.

Definition 1 (Wiener measure). *Given $\Omega = C[0,\infty)$, Borel σ-field $\mathcal{B}$ defined on $C[0,\infty)$ and any value $\sigma > 0$, a probability measure $\mathbb{P}$ satisfying the following properties is called the* Wiener measure*:*

1. $\mathbb{P}(B(0) = 0) = 1$.

2. $\mathbb{P}$ *has the independent increments property. Namely for every $0 \le t_1 < \cdots < t_k < \infty$ and $x_1, \ldots, x_{k-1} \in \mathbb{R}$,*

$$\mathbb{P}(\omega \in C[0,\infty) : B(t_2,\omega) - B(t_1,\omega) \le x_1, \ldots, B(t_k,\omega) - B(t_{k-1},\omega) \le x_{k-1})$$
$$= \prod_{2 \le i \le k} \mathbb{P}(\omega \in C[0,\infty) : B(t_i,\omega) - B(t_{i-1}),\omega) \le x_{i-1})$$

3. For every $0 \le s < t$ the distribution of $B(t) - B(s)$ is normal $N(0, \sigma^2(t-s))$. In particular, the variance is a linear function of the length of the time increment $t - s$ and the increments are stationary.

The stochastic process B described by this probability space $(C[0,\infty), \mathcal{B}, \mathbb{P})$ is called Brownian motion*. When $\sigma = 1$, it is called the* standard Brownian motion.

Theorem 2 (Existence of Wiener measure). *For every $\sigma \geq 0$ there exists a unique Wiener measure.*

(what is the Wiener measure when $\sigma = 0$?). As we mentioned before, we delay the proof of this fundamental result. For now just assume that the theorem holds and study the properties.

Remarks :

- In future we will not be explicitly writing samples ω when discussing Brownian motion. Also when we say $B(t)$ is a Brownian motion, we understand it both as a Wiener measure or simply a sample of it, depending on the context. There should be no confusion.
- It turns out that for any given σ such a probability measure is unique. On the other hand note that if $B(t)$ is a Brownian motion, then $-B(t)$ is also a Brownian motion. Simply check that all of the conditions of the Wiener measure hold. Why is there no contradiction?
- Sometimes we will consider a Brownian motion which does not start at zero: $\bar{B}(0) = x$ for some value $x \neq 0$. We may define this process as $x + B(t)$, where B is Brownian motion.

Problem 1.

1. *Let Ω be the space of all (not necessarily continuous) functions $\omega : \mathbb{R}_+ \to \mathbb{R}$.*

   i *Construct an example of a stochastic process in Ω which satisfies conditions (a)-(c) of the Brownian motion, but such that every path is almost surely discontinuous.*

   i *Construct an example of a stochastic process in Ω which satisfies conditions (a)-(c) of the Brownian motion, but such that every path is almost surely discontinuous in* every *point $t \in [0, 1]$.*

   HINT: work with the Brownian motion.

2. *Suppose $B(t)$ is a stochastic process defined on the set of all (not necessarily continuous) functions $x : \mathbb{R}_+ \to \mathbb{R}$ satisfying properties (a)-(c) of Definition 1. Prove that for every $t \geq 0$, $\lim_{n\to\infty} B(t + \frac{1}{n}) = B(t)$ almost surely.*

Problem 2. *Let $A_{\mathbb{R}}$ be the set of all homogeneous linear functions $x(t) = at$ where a varies over all values $a \in \mathbb{R}$. $B(t)$ denotes the standard Brownian motion. Prove that $\mathbb{P}(B \in A_{\mathbb{R}}) = 0$.*

4 Properties

We now derive several properties of a Brownian motion. We assume that $B(t)$ is a standard Brownian motion.

Joint distribution. Fix $0 < t_1 < t_2 < \cdots < t_k$. Let us find the joint distribution of the random vector $(B(t_1), \ldots, B(t_k))$. Given $x_1, \ldots, x_k \in \mathbb{R}$ let us find the joint density of $(B(t_1), B(t_2), \ldots, B(t_k))$ in $(x_1, \ldots, x_k)$. It is equal to the joint density of $(B(t_1), B(t_2) - B(t_1), \ldots, B(t_k) - B(t_{k-1}))$ in $(x_1, x_2 - x_1, \ldots, x_k - x_{k-1})$, which by independent Gaussian increments property of the Brownian motion is equal to

$$\prod_{i=1}^{k-1} \frac{1}{\overline{2\pi(t_{i+1} - t_i)}} e^{-\frac{(x_{i+1} - x_i)^2}{2(t_{i+1} - t_i)}}.$$

Differential Property. For any $s > 0$, $B_s(t) = B(t+s) - B(s), t \geq 0$ is a Brownian motion. Indeed $B_s(0) = 0$ and the process has independent increments $B_s(t_2) - B_s(t_1) = B(t_2 + s) - B(t_1 + s)$ which have a Guassian distribution with variance $t_2 - t_1$.

Scaling. For every c, $cB(t)$ is a Brownian motion with variance $\sigma^2 = c^2$. Indeed, continuity and the stationary independent increment properties as well as the Gaussian distribution of the increments, follow immediately. The variance of the increments $cB(t_2) - cB(t_1)$ is $c^2(t_2 - t_1)$. $\square$

For every positive $c > 0$, $B(\frac{t}{c})$ is a Brownian motion with variance $\frac{1}{c}$. Indeed, the process is continuous. The increments are stationary, independent with Gaussian distribution. For every $t_1 < t_2$, by definition of the standard Brownian motion, the variance of $B(\frac{t_2}{c}) - B(\frac{t_1}{c})$ is $(t_2 - t_1)/c = \frac{1}{c}(t_2 - t_1)$. $\square$

Combining these two properties we obtain that $\sqrt{c}B(\frac{t}{c})$ is also a standard Brownian motion.

Covariance. Fix $0 \le s \le t$. Let us compute the covariance

$$\begin{aligned} Cov(B(t), B(s)) &= \mathbb{E}[B(t)B(s)] - \mathbb{E}[B(t)]\mathbb{E}[B(s)] \\ &\stackrel{\text{a}}{=} \mathbb{E}[B(t)B(s)] \\ &= \mathbb{E}[(B(s) + B(t) - B(s))B(s)] \\ &\stackrel{\text{b}}{=} \mathbb{E}[B^2(s)] + \mathbb{E}[B(t) - B(s)]\mathbb{E}[B(s)] \\ &= s + 0 = s. \end{aligned}$$

Here (a) follows since $\mathbb{E}[B(t)] = 0$ for all t and (b) follows since by definition of the standard Brownian motion we have $\mathbb{E}[B^2(s)] = s$ and by independent increments property we have $\mathbb{E}[(B(t)-B(s))B(s)] = \mathbb{E}[(B(t)-B(s))(B(s)-B(0))] = \mathbb{E}[(B(t) - B(s)]\mathbb{E}[(B(s) - B(0)] = 0$, since increments have a zero mean Gaussian distribution.

Time reversal. Given a standard Brownian motion $B(t)$ consider the process $B^{(1)}(t)$ defined by $B^{(1)}(t) = tB(\frac{1}{t})$ for all $t > 0$ and $B^{(1)}(0) = 0$. In other words we reverse time by the transformation $t \to \frac{1}{t}$. We claim that $B^{(1)}(t)$ is also a standard Brownian motion.

Proof. We need to verify properties (a)-(c) of Definition 1 plus continuity. The continuity at any point $t > 0$ follows immediately since $1/t$ is continuous function. B is continuous by assumption, therefore $tB(\frac{1}{t})$ is continuous for all $t > 0$. The continuity at $t = 0$ is the most difficult part of the proof and we delay it till the end. For now let us check (a)-(c).

(a) follows since we defined $B^{(1)}(0)$ to be zero.

We delay (b) till we establish normality in (c)

(c) Take any $s < t$. Write $tB(\frac{1}{t}) - sB(\frac{1}{s})$ as

$$tB(\frac{1}{t}) - sB(\frac{1}{s}) = (t-s)B(\frac{1}{t}) + sB(\frac{1}{t}) - sB(\frac{1}{s})$$

The distribution of $B(\frac{1}{t}) - B(\frac{1}{s})$ is Gaussian with zero mean and variance $\frac{1}{s} - \frac{1}{t}$, since B is standard Brownian motion. By scaling property, the distribution of $sB(\frac{1}{t}) - sB(\frac{1}{s})$ is zero mean Gaussian with variance $s^2(\frac{1}{s} - \frac{1}{t})$. The distribution of $(t-s)B(\frac{1}{t})$ is zero mean Gaussian with variance $(t-s)^2(\frac{1}{t})$ and also it is independent from $sB(\frac{1}{t}) - sB(\frac{1}{s})$ by independent increments properties of the Brownian motion. Therefore $tB(\frac{1}{t}) - sB(\frac{1}{s})$ is zero mean Gaussian with variance

$$s^2(\frac{1}{s} - \frac{1}{t}) + (t-s)^2(\frac{1}{t}) = t - s.$$

This proves (c).

We now return to (b). Take any $t_1 < t_2 < t_3$. We established in (c) that all the differences $B^{(1)}(t_2) - B^{(1)}(t_1), B^{(1)}(t_3) - B^{(1)}(t_2), B^{(1)}(t_3) - B^{(1)}(t_1) = B^{(1)}(t_3) - B^{(1)}(t_2) + B^{(1)}(t_2) - B^{(1)}(t_1)$ are zero mean Gaussian with variances $t_2 - t_1, t_3 - t_2$ and $t_3 - t_1$ respectively. In particular the variance of $B^{(1)}(t_3) - B^{(1)}(t_1)$ is the sum of the variances of $B^{(1)}(t_3) - B^{(1)}(t_2)$ and $B^{(1)}(t_2) - B^{(1)}(t_1)$. This implies that the covariance of the summands is zero. Moreover, from part (b) it is not difficult to establish that $B^{(1)}(t_3) - B^{(1)}(t_2)$ and $B^{(1)}(t_2) - B^{(1)}(t_1)$ are *jointly* Gaussian. Recall, that two jointly Gaussian random variables are independent if and only if their covariance is zero.

It remains to prove the continuity at zero of $B^{(1)}(t)$. We need to show the continuity almost surely, so that the zero measure set corresponding to the samples $\omega \in C[0,\infty)$ where the continuity does not hold, can be thrown away. Thus, we need to show that the probability measure of the set

$$A = \{\omega \in C[0,\infty) : \lim_{t\to 0} tB(\frac{1}{t}, \omega) = 0\}$$

is equal to unity.

We will use Strong Law of Large Numbers (SLLN). First set $t = 1/n$ and consider $tB(\frac{1}{t}) = B(n)/n$. Because of the independent Gaussian increments property $B(n) = {}_{1\le i\le n}(B(i) - B(i-1))$ is the sum of independent i.i.d. standard normal random variables. By SLLN we have then $B(n)/n \to \mathbb{E}[B(1) - B(0)] = 0$ a.s. We showed convergence to zero along the sequence $t = 1/n$ almost surely. Now we need to take care of the other values of t, or equivalently, values $s \in [n, n+1)$. For any such s we have

$$\begin{aligned}
|\frac{B(s)}{s} - \frac{B(n)}{n}| &\le |\frac{B(s)}{s} - \frac{B(n)}{s}| + |\frac{B(n)}{s} - \frac{B(n)}{n}| \\
&\le |B(n)||\frac{1}{s} - \frac{1}{n}| + \frac{1}{n}\sup_{n\le s\le n+1}|B(s) - B(n)| \\
&\le \frac{|B(n)|}{n^2} + \frac{1}{n}\sup_{n\le s\le n+1}|B(s) - B(n)|.
\end{aligned}$$

We know from SLLN that $B(n)/n \to 0$ a.s. Moreover then

$$B(n)/n^2 \to 0. \tag{1}$$

a.s. Now consider the second term and set $Z_n = \sup_{n\le s\le n+1} |B(s) - B(n)|$. We claim that for every $\epsilon > 0$,

$$\mathbb{P}(Z_n/n > \epsilon \text{ i.o.}) = \mathbb{P}(\omega \in C[0,\infty) : Z_n(\omega)/n > \epsilon \text{ i.o.}) = 0 \qquad (2)$$

where i.o. stands for infinitely often. Suppose (2) was indeed the case. The equality means that for almost all samples ω the inequality $Z_n(\omega)/n > \epsilon$ happens for at most finitely many n. This means exactly that for almost all ω (that is a.s.) $Z_n(\omega)/n \to 0$ as $n \to \infty$. Combining with (1) we would conclude that a.s.

$$\sup_{n\le s\le n+1} |\frac{B(s)}{s} - \frac{B(n)}{n}| \to 0,$$

as $n \to \infty$. Since we already know that $B(n)/n \to 0$ we would conclude that a.s. $\lim_{s\to\infty} B(s)/s = 0$ and this means almost sure continuity of $B^{(1)}(t)$ at zero.

It remains to show (2). We observe that due to the independent stationary increments property, the distribution of Z_n is the same as that of Z_1. This is the distribution of the maximum of the absolute value of a standard Brownian motion during the interval $[0,1]$. In the following lecture we will show that this maximum has finite expectation: $\mathbb{E}[|Z_1|] < \infty$. On the other hand

$$\mathbb{E}[|Z_1|] = \int_0^\infty \mathbb{P}(|Z_1| > x)dx = \sum_{n=0}^\infty \int_{n\epsilon}^{(n+1)\epsilon} \mathbb{P}(|Z_1| > x)dx$$

$$\ge \epsilon \sum_{n=0}^\infty \mathbb{P}(|Z_1| > (n+1)\epsilon).$$

We conclude that the sum in the right-hand side is finite. By i.i.d. of Z_n

$$\sum_{n=1}^\infty \mathbb{P}(\frac{|Z_n|}{n} > \epsilon) = \sum_{n=1}^\infty \mathbb{P}(\frac{|Z_1|}{n} > \epsilon)$$

Thus the sum on the left-hand side is finite. Now we use the Borel-Cantelli Lemma to conclude that (2) indeed holds. □

5 Additional reading materials

- Sections 6.1 and 6.4 from Chapter 6 of Resnick's book "Adventures in Stochastic Processes" in the course packet.
- Durrett [2], Section 7.1
- Billingsley [1], Chapter 8.

References

[1] P. Billingsley, *Convergence of probability measures*, Wiley-Interscience publication, 1999.

[2] R. Durrett, *Probability: theory and examples*, Duxbury Press, second edition, 1996.

[3] S. Resnick, *Adventures in stochastic processes*, Birkhuser Boston, Inc., 1992.

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