Additional Homework Problems

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1 Exercises

1. Let $\mathbb{R}^n_+$ denote the nonnegative orthant, namely $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_j \geq 0, j = 1, \ldots, n\}$. Considering $\mathbb{R}^n_+$ as a cone, prove that $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$, thus showing that $\mathbb{R}^n_+$ is self-dual.

2. Let $Q^n = \left\{x \in \mathbb{R}^n \mid x_1 \geq \sqrt{\sum_{j=2}^n x_j^2}\right\}$. Q^n is called the second-order cone, the Lorentz cone, or the ice-cream cone (I am not making this up). Considering Q^n as a cone, prove that $(Q^n)^* = Q^n$, thus showing that Q^n is self-dual.

3. Prove Corollary 3 of the notes on duality theory, which asserts that the existence of Slater point for the conic dual problem guarantees strong duality and that the primal attains its optimum.

4. Consider the following "minimax" problems:

$$\min_{x \in X} \max_{y \in Y} \phi(x, y) \quad \text{and} \quad \max_{y \in Y} \min_{x \in X} \phi(x, y)$$

   where X and Y are nonempty compact convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $\phi(x, y)$ is convex in x for fixed y, and is concave in y for fixed x.

   (a) Show that $\min_{x \in X} \max_{y \in Y} \phi(x, y) \geq \max_{y \in Y} \min_{x \in X} \phi(x, y)$ in the absence of any convexity/concavity assumptions on X, Y, and/or $\phi(\cdot, \cdot)$.

   (b) Show that $f(x) := \max_{y \in Y} \phi(x, y)$ is a convex function in x and that $g(y) := \min_{x \in X} \phi(x, y)$ is a concave function in y.

   (c) Use a separating hyperplane theorem to prove:

$$\min_{x \in X} \max_{y \in Y} \phi(x, y) = \max_{y \in Y} \min_{x \in X} \phi(x, y) .$$

5. Let X and Y be nonempty sets in $\mathbb{R}^n$, and let $f(\cdot), g(\cdot) : \mathbb{R}^n \to \mathbb{R}$. Consider the following *conjugate functions* $f^*(\cdot)$ and $g^*(\cdot)$ defined as follows:

$$f^*(u) := \inf_{x \in X} \{f(x) - u^t x\} ,$$

   and

$$g^*(u) := \sup_{x \in X} \{g(x) - u^t x\} .$$

(a) Construct a geometric interpretation of $f^*(\cdot)$ and $g^*(\cdot)$.

(b) Show that $f^*(\cdot)$ is a concave function on $X^* := \{u \mid f^*(u) > -\infty\}$, and $g^*(\cdot)$ is a convex function on $Y^* := \{u \mid g^*(u) < +\infty\}$.

(c) Prove the following *weak duality theorem* between the *conjugate primal problem* $\inf\{f(x) - g(x) \mid x \in X \cap Y\}$ and the *conjugate dual problem* $\sup\{f^*(u) - g^*(u) \mid u \in X^* \cap Y^*\}$:

$$\inf\{f(x) - g(x) \mid x \in X \cap Y\} \geq \sup\{f^*(u) - g^*(u) \mid u \in X^* \cap Y^*\} \ .$$

(d) Now suppose that $f(\cdot)$ is a convex function, $g(\cdot)$ is a concave function, $\text{int}X \cap \text{int}Y \neq \emptyset$, and $\inf\{f(x) - g(x) \mid x \in X \cap Y\}$ is finite. Show that equality in part (5c) holds true and that $\sup\{f^*(u) - g^*(u) \mid u \in X^* \cap Y^*\}$ is attained for some $u = u^*$.

(e) Consider a standard inequality constrained nonlinear optimization problem using the following notation:

$$\begin{array}{llll} \text{OP}: & \text{minimum}_x & \bar{f}(x) & \\ & \text{s.t.} & \bar{g}_1(x) & \leq \ 0, \\ & & \vdots & \\ & & \bar{g}_m(x) & \leq \ 0, \\ & & x \in \bar{X} \ . & \end{array}$$

By suitable choices of $f(\cdot), g(\cdot), X$, and Y, formulate this problem as an instance of the conjugate primal problem $\inf\{f(x) - g(x) \mid x \in X \cap Y\}$. What is the form of the resulting conjugate dual problem $\sup\{f^*(u) - g^*(u) \mid u \in X^* \cap Y^*\}$?

6. Consider the following problem:

$$\begin{array}{rl} z^* = \min & x_1 + x_2 \\ \text{s.t.} & x_1^2 + x_2^2 = 4, \\ & -2x_1 - x_2 \leq 4 \ . \end{array}$$

(a) Formulate the Lagrange dual of this problem by incorporating both constraints into the objective function via multipliers u_1, u_2.

(b) Compute the gradient of $L^*(u)$ at the point $\bar{u} = (1, 2)$.

(c) Starting from $\bar{u} = (1, 2)$, perform one iteration of the steepest ascent method for the dual problem. In particular, solve the following problem where $\bar{d} = \nabla L^*(\bar{u})$:

$$\begin{array}{rl} \max_\alpha & L^*(\bar{u} + \alpha \bar{d}) \\ \text{s.t.} & \bar{u} + \alpha \bar{d} \geq 0 \ , \\ & \alpha \geq 0 \ . \end{array}$$

7. Prove Remark 1 of the notes on conic duality, that "the dual of the dual is the primal" for the conic dual problems of Section 13 of the duality notes.

8. Consider the following very general conic problem:

$$\begin{array}{lll} \text{GCP}: & z^* = \text{minimum}_x & c^T x \\ & \text{s.t.} & Ax - b \in K_1 \\ & & x \in K_2 \ , \end{array}$$

where $K_1 \subset \mathbb{R}^m$ and $K_2 \subset \mathbb{R}^n$ are each a closed convex cone. Derive the following conic dual for this problem:

$$\begin{array}{lll} \text{GCD}: & v^* = \text{maximum}_y & b^T y \\ & \text{s.t.} & c - A^T y \in K_2^* \\ & & y \in K_1^* \ , \end{array}$$

and show that the dual of GCD is GCP. How is the conic format of Section 13 of the duality notes a special case of GCP?

9. For a (square) matrix $M \in \mathbb{R}^{n \times n}$, define $\text{trace}(M) = \sum_{j=1}^{n} M_{jj}$, and for two matrices $A, B \in \mathbb{R}^{k \times l}$ define

$$A \bullet B := \sum_{i=1}^{k} \sum_{j=1}^{l} A_{ij} B_{ij} \ .$$

Prove that:

(a) $A \bullet B = \text{trace}(A^T B)$.

(b) $\text{trace}(MN) = \text{trace}(NM)$.

10. Let $S_+^{k\times k}$ denote the cone of positive semi-definite symmetric matrices, namely $S_+^{k\times k} = \{X \in S^{k\times k} \mid v^T X v \geq 0 \text{ for all } v \in \Re^n\}$. Considering $S_+^{k\times k}$ as a cone, prove that $\left(S_+^{k\times k}\right)^* = S_+^{n\times n}$, thus showing that $S_+^{k\times k}$ is self-dual.

11. Consider the problem:

$$
\begin{array}{llcccl}
\text{P}: \quad z^* = \text{minimum}_{x_1,x_2,x_3} & x_1 & & & & \\
\text{s.t.} & & x_2 & +x_3 & = & 0 \\
 & -x_1 & & & \leq & 10 \\
 & \|(x_1,x_2)\| \leq x_3 \quad , & & & &
\end{array}
$$

where $\|\cdot\|$ denotes the Euclidean norm. Then note that this problem is feasible (set $x_1 = x_2 = x_3 = 0$), and that the first and third constraints combine to force $x_1 = 0$ in any feasible, solution, whereby $z^* = 0$.

We will dualize on the the first two constraints, setting

$$X := \{(x_1, x_2, x_3) \mid \|(x_1, x_2)\| \leq x_3\} \ .$$

Using multipliers u_1, u_2 for the first two constraints, our Lagrangian is:

$$L(x_1, x_2, x_3, u_1, u_2) = x_1 + u_1(x_2 + x_3) + u_2(-x_1 - 10) = -10u_2 + (1 - u_2, u_1, u_1)^T (x_1, x_2, x_3) \ .$$

Then

$$L^*(u_1, u_2) = \min_{\|(x_1,x_2)\| \leq x_3} \ -10u_2 + (1 - u_2, u_1, u_1)^T (x_1, x_2, x_3) \ .$$

(i) Show that:

$$L^*(u_1, u_2) = \begin{cases} -10u_2 & \text{if} \quad \|(1 - u_2, u_1)\| \leq u_1 \\ -\infty & \text{if} \quad \|(1 - u_2, u_1)\| > u_1 \ , \end{cases}$$

and hence the dual problem can be written as:

$$
\begin{array}{llll}
\mathrm{D}: & v^* = \text{maximum}_{u_1,u_2} & & -10u_2 \\
& & \text{s.t.} & \|(1-u_2,u_1)\| \le u_1 \\
& & & u_1 \in \mathbb{R}, u_2 \ge 0 \ .
\end{array}
$$

(ii) Show that $v^* = -10$, and that the set of optimal solutions of D is comprised of those vectors $(u_1, u_2) = (\alpha, 1)$ for all $\alpha \ge 0$. Hence both P and D attain their optima with a finite duality gap.

(iii) In this example the primal problem is a convex problem. Why is there nevertheless a duality gap? What hypotheses are absent that otherwise would guarantee strong duality?