Practice Problems

1. Consider a 3-period model with $t = 0, 1, 2, 3$. There are a stock and a risk-free asset. The initial stock price is \$4 and the stock price doubles with probability 2/3 and drops to one-half with probability 1/3 each period. The risk-free rate is 1/4.

   (a) Compute the risk-neutral probability at each node.

   (b) Compute the Radon-Nikodym derivative ($d\mathbf{Q}/d\mathbf{P}$) of the risk-neutral measure with respect to the physical measure at each node.

   (c) Compute the state-price density at each node.

   (d) Price a lookback option with payoff at $t = 3$ equal to $(\max_{0 \le t \le 3} S_t) - S_3$ using risk-neutral probability.

   (e) Price the lookback option using state-price density and compare your answer to (d).

2. Show that, under the risk-neutral measure, the discounted gain process

$$\hat{G}_t = \frac{P_t}{B_t} + \sum_{s=1}^{t} \frac{D_s}{B_s}$$

   is a martingale (i.e. $E_t^Q\left[\hat{G}_{t+1}\right] = \hat{G}_t$) from the definition of risk-neutral measure in lecture notes

$$P_t = E_t^Q\left[\sum_{u=t+1}^{T} \frac{B_t}{B_u} D_u\right]$$

   That is the reason why the risk-neutral measure is also called the "equivalent martingale measure" (EMM).

3. Consider the following model of interest rates. Under the physical probability measure $\mathbf{P}$, the short-term interest rate is $\exp(r_t)$, where r_t follows

$$dr_t = -\theta(r_t - \overline{r})\, dt + \sigma_r dZ_t,$$

   where Z_t is a Brownian motion.

   Assume that the SPD is given by

$$\pi_t = \exp\left(-\int_0^t r_u + \frac{1}{2}\eta_u^2\, du - \int_0^t \eta_u\, dZ_u\right)$$

where η_t is stochastic, and follows

$$d\eta_t = -\kappa(\eta_t - \overline{\eta})\, dt + \sigma_\eta\, dZ_t^\eta$$

where Z_t^η is a Brownian motion independent of Z_t.

(a) Derive the dynamics of the interest rate under the risk-neutral probability $\mathbf{Q}$.

(b) Compute the spot interest rates for all maturities. (Hint: look for bond prices in the form $P(t,T) = \exp(a(T-t) + b(T-t)r_t + c(T-t)\eta_t))$.

(c) Compute the instantaneous expected rate of return on a zero-coupon bond with time to maturity τ.

(d) Show that the slope of the term structure of interest rates predicts the excess returns on long-term bonds. Discuss the intuition. Show that more volatility in the price of risk, η, means more predictability in bond returns.

4. Suppose that uncertainty in the model is described by two independent Brownian motions, $Z_{1,t}$ and $Z_{2,t}$. Assume that there exists one risky asset, paying no dividends, following the process

$$\frac{dS_t}{S_t} = \mu(X_t)\, dt + \sigma\, dZ_{1,t}$$

where

$$dX_t = -\theta X_t\, dt + dZ_{2,t}$$

The risk-free interest rate is constant at r.

(a) What is the price of risk of the Brownian motion $Z_{1,t}$?

(b) Give an example of a valid SPD in this model.

(c) Suppose that the price of risk of the second Brownian motion, $Z_{2,t}$, is zero. Characterize the SPD in this model.

(d) Derive the price of a European Call option on the risky asset in this model, with maturity T and strike price K.

5. Consider a European call option on a stock. The stock pays no dividends and the stock price follows an Ito process. Is it possible that, while the stock price declines between t_1 and $t_2 > t_1$, the price of the Call increases? Justify your answer.

6. Suppose that the stock price S_t follows a Geometric Brownian motion with parameters μ and σ. Compute

$$\mathrm{E}_0\left[(S_T)^\lambda\right].$$

7. Suppose that, under **P**, the price of a stock paying no dividends follows

$$\frac{dS_t}{S_t} = \mu(S_t)\,dt + \sigma(S_t)\,dZ_t$$

Assume that the SPD in this market satisfies

$$\frac{d\pi_t}{\pi_t} = -r\,dt - \eta_t dZ_t$$

   (a) How does η_t relate to r, μ_t, and σ_t?

   (b) Suppose that there exists a derivative asset with price $C(t, S_t)$. Derive the instantaneous expected return on this derivative as a function of t and S_t.

   (c) Derive the PDE on the price of the derivative $C(t, S)$, assuming that its payoff is given by $H(S_T)$ at time T.

   (d) Suppose that there is another derivative trading, with a price $D(t, S_t)$ which does not satisfy the PDE you have derived above. Construct a trading strategy generating arbitrage profits using this derivative, the risk-free asset and the stock.

8. Consider a futures contract with price changing according to

$$F_{t+1} = F_t + \lambda + \mu_t + \sigma_F \varepsilon_t,$$
$$\mu_{t+1} = \rho\mu_t + \sigma_\mu u_t$$

where ε_t and u_t are independent IID $\mathcal{N}(0,1)$ random variables. Assume that the interest rate is constant at r. Your objective is to construct an optimal strategy of trading futures between $t = 0$ and T to maximize the terminal objective

$$\mathrm{E}\left[-e^{-\alpha W_T}\right]$$

where W_T is the terminal value of the portfolio. Assume the initial portfolio value of W_0.

   (a) Formulate the problem as a dynamic program. Describe the state vector, verify that it follows a controlled Markov process.

   (b) Derive the value function at T and $T-1$ and optimal trading strategy at $T-1$ and $T-2$.

9. Suppose you can trade two assets, a risk-free bond with interest rate r and a risky stock, paying no dividends, with price S_t. Assume $S_{t+1} = S_t \times \exp(\mu + \sigma\varepsilon_t)$ where ε_t are IID $\mathcal{N}(0,1)$ random variables.

   Assume that whenever you buy the stock you must pay transaction costs, but you can sell stock without costs. Specifically, when you buy X dollars worth of stock, you must pay $(1+\tau)X$, so the fee is proportional, given by τ. Your objective is to figure out how to trade optimally to maximize the objective

$$\mathrm{E}\left[-e^{-\alpha W_T}\right]$$

where W_T is the terminal value of the portfolio.

(a) What should be the state vector for this problem? Formulate the problem as a dynamic program, verify the assumptions on the state vector and the payoff function.

(b) Write down the Bellman equation.

10. Suppose we observe returns on N independent trading strategies, r_t^n, $n = 1, 2$, $t = 1, ..., T$. Assume that returns are IID over time, and each strategy has normal distribution:

$$r_t^n \sim \mathcal{N}(\mu_n, \sigma^2)$$

Assume $\mu_1 > \mu_2$.

(a) Estimate the mean return on each strategy by maximum likelihood. Express $\widehat{\mu}_n$ as a function of observed returns on strategy n.

(b) Since returns are normally distributed, $\widehat{\mu}_n$ is also normally distributed. Describe its distribution. (In general, for arbitrary return distribution, $\widehat{\mu}_n$ is only approximately normal).

(c) What is the distribution of $\max_n(\widehat{\mu}_n)$? characterize it using the CDF function.

(d) Suppose you are interested in identifying the strategy with the higher mean return. You pick the strategy with the higher estimated mean. What is the probability that you have made a mistake?

11. Suppose interest rate follows an AR(1) process

$$r_t - \overline{r} = \theta(r_{t-1} - \overline{r}) + \varepsilon_t$$

where ε_t are IID $\mathcal{N}(0, \sigma^2)$ random variables. You want to estimate the average rate, $\overline{r}$, based on the sample r_t, $t = 0, 1, ..., T$. Assume that we know the true value of θ.

(a) Derive the estimate of $\overline{r}$ by maximum likelihood.

(b) Show that this estimate is valid even if the shocks ε_t are not normally distributed, as long as the mean of ε_t is zero.

(c) Treating ε_t as IID, derive the asymptotic variance of your estimator of $\overline{r}$. Do not use Newey-West, derive the result from first principles. How does the answer depend on θ?

12. Suppose you observe two time series, X_t and Y_t. You have a model for Y_t:

$$Y_{t+1} = \rho Y_t + (a_0 + a_1 X_t)\,\varepsilon_{t+1},\ \ t = 0, 1, ..., T$$

where $\varepsilon_{t+1} \sim \mathcal{N}(0,1)$, IID. Assume that the shocks ε_t are independent of the process X_t and the lagged values of Y_t. There is no model for X_t.

(a) Using the GMM framework, which moment condition can be used to estimate ρ?

(b) Argue why it is valid to estimate ρ using an OLS regression of Y_{t+1} on Y_t.

(c) Suppose that the variance of the estimator $\widehat{\rho}$ is $(1/T)\sigma_\rho^2$. Describe how you would test the hypothesis that $\rho = 0$.

(d) Write down the conditional log-likelihood function $\mathcal{L}(\rho, a_0, a_1)$.

(e) Suppose that the parameters a_0 and a_1 are known. Derive the maximum-likelihood estimate for ρ.

13. Suppose we observe a sequence of IID random variables $X_t \geq 0$, $t = 1, ..., T$, with probability density

$$pdf(X) = \lambda e^{-\lambda X}, \quad X \geq 0$$

(a) Write down the log-likelihood function $\mathcal{L}(\lambda)$.

(b) Compute the maximum likelihood estimate $\widehat{\lambda}$.

(c) Derive the standard error for $\widehat{\lambda}$.

14. Suppose you observe a series of observations X_t, $t = 1, ..., T$. You need to fit a model

$$X_{t+1} = f(X_t, X_{t-1}; \theta) + \varepsilon_{t+1}$$

where $\mathrm{E}[\varepsilon_{t+1}|X_t, X_{t-1}, ..., X_1] = 0$. Innovations ε_{t+1} have zero mean conditionally on X_t, X_{t-1},...,X_1. You also know that innovations ε_{t+1} have constant conditional variance:

$$\mathrm{E}[\varepsilon_{t+1}^2|X_t, X_{t-1}, ..., X_1] = \sigma^2$$

The parameter σ is not known. θ is the scalar parameter affecting the shape of the function $f(X_t, X_{t-1}; \theta)$.

(a) Describe how to estimate the parameter θ using the quasi maximum likelihood approach. Derive the relevant equations.

(b) Describe in detail how to use parametric bootstrap to estimate a 95% confidence interval for θ.

(c) Describe how to estimate the bias in your estimate of θ using parametric bootstrap.

(d) Derive the asymptotic standard error for $\widehat{\theta}$ (large T) using GMM standard error formulas.

15. Consider an estimator $\widehat{\theta}$ for a scalar-valued parameter θ. Suppose you know, as a function of the true parameter value θ_0, the distribution function of the estimator, i.e., you know

$$CDF_{\widehat{\theta}-\theta_0}(x)$$

(In practice, you may be able to estimate the above CDF using bootstrap). Note that this CDF does not depend on model parameters.

Based on the definition of the confidence interval, derive a formula for a confidence interval which covers the true parameter value with probability 95%.

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